Contest Architecture with Public Disclosures*

Preliminary and Incomplete.

Toomas Hinnosaar†

May 5, 2019‡

Abstract

I study optimal sequential contests, where contestants arrive over time and the contest designer chooses a disclosure rule which specifies at which periods to publicly disclose the efforts of the previous contestants. I find optimal contests for a wide range of possible objectives. While the set of all potential contests is large and different objectives involve different trade-offs, I show that most standard objective functions are maximized by the same three contest structures that are widely studied in the literature: simultaneous, first-mover, or sequential contests.

JEL: C72, C73, D72, D82
Keywords: contests, sequential games, contest design, rent-seeking, R&D, advertising

1 Introduction

In this paper, I study sequential contests and contest-like economic interactions, where the payoffs of the participants are increasing in their own efforts and decreasing in the total effort. For example, in a typical rent-seeking contest, lobbyists are choosing costly efforts and the probability of achieving a favorable outcome is proportional to their efforts. Similarly, firms entering oligopolistic markets invest to increase capacity and the market price decreases with the total capacity. Players make these choices over time and depending on the disclosure policy they may have more or less information about the choices the earlier-movers made. For example, a fully transparent lobbying disclosure rule would lead to sequential rent-seeking effort choices, whereas a non-transparent policy may lead to more independent choices of lobbying efforts.[1] Many economic interactions satisfy the

*I would like to thank Federico Boffa, Juan Carlos Carbajal, Dino Gerardi, Marit Hinnosaar, Ignacio Monzón, and Emil Temnyalov for their comments and suggestions.

†Collegio Carlo Alberto, `toomas@hinnosaar.net`.

‡First draft: May 13, 2018. The latest version: `toomas.hinnosaar.net/contest_architecture.pdf`.

[1]Different countries have adopted different transparency rules regulating lobbying. For example, in the United States lobbying efforts are all recorded and reported quarterly (Lobbying Disclosure Act, 1995; Honest Leadership and Open Government Act, 2007), whereas in the European Union reporting is arranged on a more voluntary basis and on a yearly frequency (European Transparency Initiative, 2005).

assumptions of the model discussed here, including oligopolies, public goods provision, rent-seeking, research and development, advertising, and sports.

In different situations, the contest designer may have different objectives. Often the goal is to maximize the total effort (such as research and development efforts) or minimize the total effort (such as rent-seeking efforts). However, there are also many situations where other objectives are more natural. For example, schools designing incentives for students to learn are typically not maximizing the average effort, but rather making sure that all students learn.[2] Similarly, in firms where efforts are complementary it is often most important to incentivize the lowest-performing employees.[3] In welfare economics, there are two standard assumptions. The utilitarian social planner maximizes the total welfare of the participants, whereas the Rawlsian social planner maximizes the lowest utility. Finally, we can also think of the contest designer's problem in a larger context, where the results provide bounds. For example, in studying entry to a rent-seeking contest we may want to make sure that none of the contestants expects to achieve too high payoff to make sure that there is no excessive entry. Therefore we would be interested in the minimum of the highest payoff.

In this paper, I study optimal contest under various natural objectives of the contest designer. In particular, I consider minimizing and maximizing eight different objectives: total effort, total welfare, lowest effort, lowest payoff, highest effort, highest payoff, effort inequality, and payoff inequality. On one hand, the goal is to understand what types of contest structures are optimal in different problems and why, and to provide a menu of results that researchers and practitioners could build upon. On the other hand, seeing a pattern in the types of contests that arise as an optimal solution to different problems allows us to understand why we see these contests used so often.

The main difficulty in studying these types of problems is tractability. Sequential games with non-quadratic payoffs are typically difficult to solve, since best-response functions are non-linear and therefore the standard backward-induction approach leads to increasingly complex expressions. This problem is even more pronounced in contest design, where the goal is to compare all possible contests. In this paper, I am building on recent progress in aggregative games (Jensen, 2010; Martimort and Stole, 2012; Acemoglu and Jensen, 2013) and sequential contests (Hinnosaar, 2018, 2019), which allows overcome these issues.

The main and perhaps surprising finding of the paper is that almost all natural objective functions of the contest designer are maximized by the same three standard types of contests. First, the *simultaneous contest*, which is the least informative form of contest that arises whenever contestants do not learn anything about their competitors' choices. This is by far the most common assumption in the literature, starting with Cournot (1838) in oligopoly theory and Tullock (1967, 1974) in contest theory. The second type of contest that is optimal in many cases is the *sequential contest*, which arises when all effort choices are public. It has been studied by Robson (1990) in the case of large oligopolies, Glazer and Hassin (2000); Hinnosaar (2018); Kahana and Klunover (2018) in case of sequential Tullock contests, and Hinnosaar (2018) with more general payoff functions. The third type is the *first-mover* contest, where a single first-mover chooses the effort first and the

[2]In fact, the education reform in the US was called No Child Left Behind Act of 2001.

[3]Referred to as the *bottleneck* or the *weakest link* in the business terminology.

rest of the players move sequentially in the second period. In oligopoly theory, it was first studied by von Stackelberg (1934) and in contest theory by Dixit (1987).[4]

The paper contributes to both contest design and information design literature. Earlier papers on contest design Glazer and Hassin (1988); Taylor (1995); Che and Gale (2003); Moldovanu and Sela (2001, 2006); Olszewski and Siegel (2016); Bimpikis et al. (2019) have focused on contests with private information and have studied which prize structure to offer and how to arrange contests into subcontests to maximize either total effort or highest effort. In this paper, I study contest design with full information (about the state and payoffs), but the designer can choose to disclose less or more information about players' efforts. There is growing literature on information design (Kamenica and Gentzkow, 2011; Bergemann and Morris, 2019), which has mostly focused on revealing information about the state of the world or private information. This paper is more in line with recent papers that have also studied disclosure of the actions of other players (Doval and Ely, 2019; Ely and Szydlowski, 2019).

The paper is organized as follows. Section 2 describes the model. Section 3 provides all results, describing the contests that minimize and maximize eight different objective functions. Section 4 summarizes and concludes. Proofs are in appendix A.

2 Model

There is a contest designer and a finite number n players (contestants), $\mathcal{N} = \{1, \ldots, n\}$. Each player i chooses effort level $x_i \geq 0$. The contest designer can choose a partition of players $\boldsymbol{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_T)$ or equivalently a vector $\mathbf{n} = (n_1, \ldots, n_T)$, where $n_t = \#\mathcal{I}_t$. This can be interpreted as players arriving sequentially and the designer choosing a disclosure rule: at which points of time the efforts (or the cumulative effort) is publicly disclosed. Alternatively, it can be interpreted as the planner dividing players between periods. The disclosure rule is chosen before effort choices and is commonly known. A player $i \in \mathcal{I}_t$ is said to arrive in period t, observes the cumulative effort of all players arriving in earlier periods $X_{t-1} = \sum_{s=1}^{t-1} \sum_{j \in \mathcal{I}_s} x_j$.

For a given profile of efforts is $\mathbf{x} = (x_1, \ldots, x_n)$ the total effort is $X = X_T = \sum_{i=1}^{n} x_i$ and the payoff of player i is $u_i(\mathbf{x}) = x_i h(X)$, where $h(X)$ is the marginal benefit of effort, which is a smooth and strictly decreasing function of the total effort X, such that $h(\overline{X}) = 0$ for some saturation point $\overline{X} \in (0, \infty)$. For example, in the case of Tullock contest $h(X) = \frac{v}{X} - c$, where v is the total value of prizes and c marginal cost of effort, and $\overline{X} = \frac{v}{c}$ is the full-dissipation level of total effort. In the case of oligopoly, $h(X) = P(X) - c$, where $P(X)$ is the inverse demand function, and $\overline{X} = P^{-1}(c)$ is the competitive quantity.

I am looking for pure-strategy subgame-perfect Nash equilibria. An equilibrium strategy profile is denoted by $\mathbf{x}^*$ and the corresponding total equilibrium effort by X^*. I assume that the two regularity assumptions from Hinnosaar (2018) hold. These conditions put some additional restrictions on the smoothness and the curvature of the function $h(X)$ and guarantee that an equilibrium exists, is unique, and is in the interior of the interval $[0, \overline{X}]$. These regularity assumptions are satisfied for many natural functional forms, including Tullock contest and oligopoly with linear demand $P(X) = a(\overline{X} - X)$.

[4] For literature reviews on dynamic contests, see Konrad (2009) and Vojnović (2015).

To compare different contests, I define *informativeness* as a partial order on all n-player contests that can be achieved by public disclosures. Contest $\widehat{\mathbf{n}}$ is more informative than contest $\mathbf{n}$, if the corresponding partition $\widehat{\boldsymbol{\mathcal{I}}}$ is a finer than $\boldsymbol{\mathcal{I}}$. Equivalently, a more informative contest $\widehat{\mathbf{n}}$ can be achieved by adding public disclosures to a less informative contest $\mathbf{n}$.

In particular, the *simultaneous contest* $\mathbf{n} = (n)$ that does not provide any information about the efforts of other players is less informative than any other contest. In the other extreme, the *sequential contest* $\mathbf{n} = (1, 1, \ldots, 1)$ discloses efforts after each player and is therefore more informative than any other contest. Finally, the *first-mover contest* $\mathbf{n} = (1, n-1)$ discloses information after the first player, whereas all other players make their moves simultaneously. The first-mover contest is less informative than any other *single-leader* contest $\mathbf{n} = (1, n_1, \ldots, n_T)$.

3 Optimal Contests

In this section, I find optimal contests that minimize or maximize the eight objective functions mentioned in the introduction. In nine out of sixteen cases I show that the optimal contest is always one of the two extremes.[5] In the remaining five cases, where the general characterization for arbitrary payoff functions is not possible, I provide three types of results. First, I derive qualitative properties that hold for any n and any $h(X)$. Secondly, I describe the optimal contests for Tullock contests with $n \in \{2, \ldots, 12\}$ players. This is a prominent and non-trivial functional form that allows me to describe the trade-offs, compute exact equilibrium outcomes, and illustrate the results with figures.[6] And third, I derive the optimal contest for the cases where either $h(X)$ is linear or n is large. Note that when n is large enough, then any smooth $h(X)$ function can be arbitrarily closely approximated with a linear function, so the results in these two cases always coincide.

3.1 Total Effort and Total Welfare

How does the total equilibrium effort X^* change with additional disclosures? Hinnosaar (2018) showed that the total effort is strictly increasing in the informativeness of the contest, i.e. each additional disclosure increases the total equilibrium effort X^*. Therefore the simultaneous contest minimizes and the sequential contest maximizes the total equilibrium effort.

The intuition for this result comes from the discouragement effect. Efforts in contests are strategic substitutes. Therefore, with each additional public disclosure, each earlier-mover whose effort is now made visible to some additional later-movers has an additional incentive to increase the effort—it discourages these later-movers to exert high effort. Moreover, in equilibrium this effect must be less than one-to-one, i.e. the decrease in

[5]For any function $h(X)$ satisfying regularity conditions and any number of players $n \in \mathbb{N}$.

[6]Without loss of generality, I normalize the prize and marginal cost to 1, so that the payoff of player i is $u_i(\mathbf{x}) = \frac{x_i}{X} - x_i$. For each n, there are $n-1$ possible points of disclosure, which means 2^{n-1} possible contests. In numerical calculations I consider $\sum_{n=2}^{12} 2^{n-1} = 4094$ possible contests with 2 to 12 players. I use the Matlab code available at `toomas.hinnosaar.net/contests/`.

efforts by later-movers is smaller than the increase in efforts by earlier-movers. Otherwise, a marginal increase in the effort x_i of an earlier-mover i would decrease the total effort X and therefore increase i's payoff. This would be a profitable deviation and thus violate equilibrium conditions.

Let us now look at total equilibrium welfare. For example, in the case of a normalized Tullock contest the total welfare is $W(\mathbf{x}^*) = \sum_{i=1}^n u_i(\mathbf{x}^*) = 1 - X^*$. This expression is clearly decreasing in X^*, so maximized by the simultaneous contest and minimized by the sequential contest. The intuition is simple—in Tullock contest, the total prize is fixed. Additional players and additional closures lead to higher efforts, which are costly and therefore welfare-reducing. The following proposition shows that these conclusions generalize to arbitrary payoffs and an arbitrary number of players.

Proposition 1 (Total Effort and Total Welfare)**.** *Total equilibrium effort X^* is minimized by the simultaneous contest and maximized by the sequential contest. Total equilibrium welfare $W(\mathbf{x}^*) = \sum_{i=1}^n u_i(\mathbf{x}^*)$ is minimized by the sequential contest and maximized by the simultaneous contest.*

3.2 Lowest Effort and Lowest Payoff

We need to first determine who is the player choosing the lowest effort in equilibrium and who gets the lowest payoff. The answer comes from earlier-mover advantage result from Hinnosaar (2018)—equilibrium efforts and payoffs of earlier-movers who are observed by more players are higher than the corresponding efforts and payoffs of later-movers. Therefore player n is choosing the lowest effort and gets the lowest payoff with any n-player contest $\mathbf{n}$. The reason for this result is again the discouragement effect—earlier players choose higher efforts to discourage later players from exerting effort whenever their efforts are made public. By doing so, earlier players ensure higher and later players get lower payoffs.

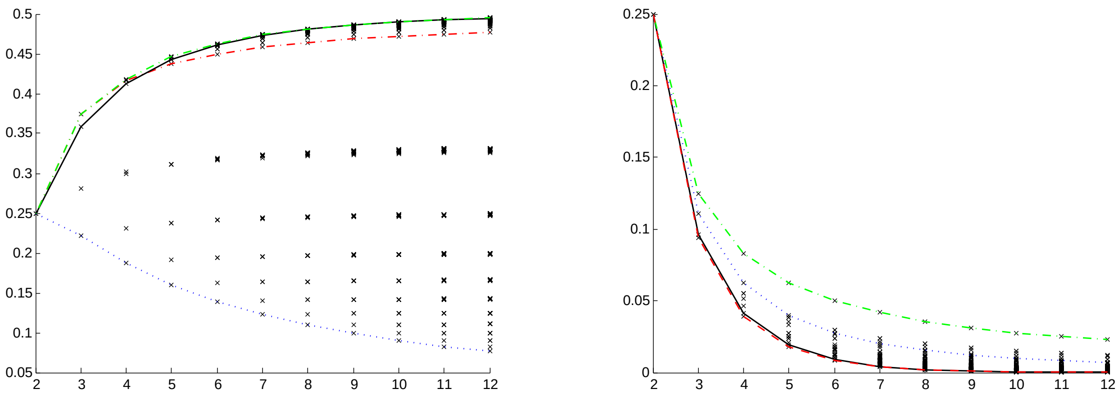
The following proposition shows that both the lowest effort and the lowest payoff are minimized and maximized by the same contests as the total welfare—sequential and simultaneous contests respectively. The intuition is simple. As discussed above, each additional disclosure leads to higher efforts by earlier-movers and lower efforts by later-movers. Therefore the discouragement effect to the last player n is increased with each disclosure. Moreover, additional disclosures lead to higher total equilibrium effort X^*, which reduces the payoff of the last player even further.

Proposition 2 (Lowest Effort and Lowest Payoff)**.** *Lowest equilibrium effort* $\min_i\{x_i^*\}$ *and lowest equilibrium payoff* $\min_i\{u_i(\mathbf{x}^*)\}$ *are minimized by the sequential contest and maximized by the simultaneous contest.*

3.3 Highest Effort and Highest Payoff

The earlier-mover advantage result shows that the player who exerts the highest effort and gets the highest payoff is the first player. It is easy to see that the effort of player 1 is minimized by the simultaneous contest. This is the contest, which minimizes the total effort X^* and where all efforts are equal. In all other contests, the total effort is higher and the effort of player 1 is higher than the average, due to the discouragement effect, which means higher than in the simultaneous contest. Indeed, this is what figure 1a confirms—for each n, the highest equilibrium effort is minimized by the simultaneous contest.

Maximization of the highest effort x_1^* is the first objective, where the optimal solution is not always one of the extremes, i.e. simultaneous or sequential contest. For example, let us consider three-player Tullock contests with a single leader. There are two such contests. The first-mover contest $\mathbf{n} = (1, 2)$ has a total equilibrium effort $X^* = 0.75$ and the highest effort $x_1^* = 0.375$. Adding a disclosure after player 2, makes the contest sequential, i.e. $\widehat{\mathbf{n}} = (1, 1, 1)$. As argued above, this raises the total effort to $\widehat{X}^* \approx 0.7887$. This reduces the marginal benefit of effort for all players. However, it also makes the discouragement effect of player 1 stronger through indirect influence. Note that players 2 and 3 already observed the effort of player 1, so direct discouragement effect is unchanged. However, in the sequential contest player 3 observes player 2, who is influenced by player 1. In the case of Tullock contest payoffs the first effect dominates and the leader's equilibrium effort reduces to $\widehat{x}_1^* \approx 0.3591 < x_1^*$. Numerical calculations summarized by figure 1a show that the contest that maximizes the highest equilibrium effort in Tullock contest with $2 < n \leq 12$ players is always one with a single leader and followers that are arranged pairwise, i.e. $\mathbf{n} = (1, 2, \ldots, 2)$ when n is odd or $\mathbf{n} = (1, 2, \ldots, 2, 1)$ when n is even.[7] Arranging the followers pairwise leads to a strictly higher effort by the first player than in any other contest, including the sequential contest.



(a) Highest effort. Green dash-dotted line indicates the single-leader contest with pairwise followers. The slightly lower red dash-dotted line indicates the first-mover contest.

(b) Highest payoff. Green dash-dotted line indicates contests $\mathbf{n} = (1, n-1)$ and red long-dashed line indicates contests $(2, 1, \ldots, 1)$.

Figure 1: Highest effort $\max_i\{x_i^*\}$ and highest payoff $\max_i\{u_i(\mathbf{x}^*)\}$ in all normalized Tullock contests with $2 \leq n \leq 12$ players. The solid black line indicates sequential contests and dashed blue line simultaneous contests.

The observation that there should be a single first-mover is general and intuitive. Making first-movers' effort observable to all players increases total effort only because increased effort by the first player, so there is no trade-off. As discussed above, each additional disclosure has two opposite effects and the effect to highest effort depends on

[7]The location of the second single-player period does not affect x_1^*.

the weights to direct and indirect influences. For example, with exponential function $h(X) = \frac{1}{\log 2}\left[2^{-X} - 2^{-1}\right]$ the first-mover contest $\mathbf{n} = (1, 2)$ gives highest effort $x_1^* \approx 0.3698$, whereas the sequential contest $\widehat{\mathbf{n}} = (1, 1, 1)$ gives strictly larger highest effort $\widehat{x}_1^* \approx 0.3714 > x_1^*$. The literature on sequential oligopolies (starting from (Daughety, 1990)) has found that if the demand function is linear, then Stackelberg leaders behave as if there are no followers, i.e. x_1^* would be independent on the number and arrangement of followers. Hinnosaar (2019) provides the most general formulation of this Stackelberg independence result, but also shows that this result holds only when $h(X)$ function is linear. Therefore the formal statement, in this case, provides only a qualitative result, showing that $n_1 = 1$. Indeed, as any single-leader is optimal in the case of linear $h(X)$, the result cannot exclude any such contest from being optimal.

Finally, in the case when n is large enough, we can say more regardless of the exact details of the payoff function. Namely, in this case, X^* is close to its upper bound and therefore assuming that $h(X)$ function is smooth, it can be closely approximated by a linear function. In particular, this means that the conclusion from linear demand extends here—any large single-leader contest leads to the highest effort that is arbitrarily close to the maximized highest effort. Figure 1a also illustrates that while all single-leader contests are optimal in the limit, for the sequential contest the convergence is very fast, whereas for the first-mover contest the convergence is much slower.[8]

Proposition 3 (Highest Effort)**.** *The highest effort* $\max_i\{x_i^*\}$ *is minimized by simultaneous contest. If contest* $\mathbf{n}$ *maximizes the highest effort then* $n_1 = 1$*. If* $h(X)$ *is linear or* n *is large enough, then any single-leader contest ensures highest effort that is arbitrarily close to maximum.*

Maximizing highest equilibrium payoff $u_1(\mathbf{x}^*)$ balances a similar trade-off, but the downward force through reduced marginal benefit $h(X^*)$ is more pronounced. Not only it pushes effort down, but it also reduces $h(X^*)$, which directly reduces all payoffs. This means that we would naturally expect the optimal contest to have fewer disclosures than the contest that maximizes the highest equilibrium effort. Indeed, figure 1b shows that among all Tullock contests with up to 12 players, the first-mover contest $\mathbf{n} = (1, n-1)$ is always optimal.

Proposition below formalizes this qualitative property—a contest that maximizes the highest equilibrium effort must have a single leader and has to be (weakly) less informative than a contest that maximizes the highest payoff. For example, in the case of linear $h(X)$ function, one maximizer of the highest effort is the first-mover contest $\mathbf{n} = (1, n-1)$. As all other contests with a single leader are more informative, the immediate conclusion is that the first-mover contest is the unique maximizer of the highest payoff. In the case of linear $h(X)$ function or a large number of players, we can therefore uniquely determine the optimal contest. It is always $\mathbf{n} = (1, n-1)$, which confirms the finding from figure 1b.

Minimizing the highest payoff $u_1(\mathbf{x}^*)$ requires balancing a different trade-off. As we saw above, additional disclosures lead to larger highest effort due to stronger discouragement effect. This would imply that minimizing the highest payoff requires relatively uninformative contest. On the other hand, additional disclosures lead to a higher total

[8]The same arguments can be used to find the contest that minimizes or maximizes the i-th highest effort or payoff when $h(X)$ is linear or n is large enough. Appendix C provides these results.

effort, which reduces all payoffs. Let us compare for example three-player Tullock contests $\mathbf{n} = (2,1)$ and $\widehat{\mathbf{n}} = (1,1,1)$. Contest $\mathbf{n}$ gives total effort $X^* = 0.75$, highest effort $x_1^* \approx 0.2813$ and therefore highest payoff $u_1(\mathbf{x}^*) \approx 0.0938$. Contest $\widehat{\mathbf{n}}$ gives total effort $\widehat{X}^* \approx 0.7887 > X^*$, but highest effort $\widehat{x}_1^* \approx 0.3591$, which is much larger than x_1^* and therefore highest payoff $u_1(\widehat{\boldsymbol{x}}^*) = 0.0962 > u_1(\mathbf{x}^*)$. Figure 1b shows that in case of Tullock contest with up to 12 players, contest $\mathbf{n} = (2,1,\ldots,1)$ minimizes the highest payoff.

The formal result below provides only one general qualitative property for the contest that minimizes the highest payoff: the number of players in the first period must be weakly higher than in any other period, i.e. $n_1 \geq n_t$ for all t. If this would not be the case, the designer could rearrange the groups while keeping the total effort unchanged and decreasing the highest effort, which would reduce the highest payoff.[9] Why can't we say more? Other possible manipulations of the contests, such as moving a player from period 1 to period 2 or splitting up the period by an additional disclosure, lead to two opposite effects: on one hand they increase the total effort in the contest, which reduces all efforts as well as the payoffs directly, but on the other hand they increase the influence that the first player has through the discouragement effects. The relative magnitudes of these effects depend on the shape of the function $h(X)$.

In the case of linear $h(X)$ or large contests, the optimal contest is again uniquely determined: it is always $(2,1,\ldots,1)$.[10] This confirms the finding on figure 1b.

Proposition 4 (Highest Payoff)**.** *If contest $\mathbf{n} = (n_1,\ldots,n_T)$ minimizes the highest equilibrium payoff $\max_i\{u_i(\mathbf{x}^*)\}$, then $n_1 \geq n_t$ for all t. If $h(X)$ is linear or n is large enough, then the optimal contest is $\mathbf{n} = (2,1,1,\ldots,1)$.*

If contest $\mathbf{n}$ maximizes the highest equilibrium payoff, then $n_1 = 1$ and $\mathbf{n}$ is not more informative than any maximizer of the highest effort. If $h(X)$ is linear or n is large enough, then the optimal contest is $\mathbf{n} = (1, n-1)$.

3.4 Effort Inequality and Payoff Inequality

In this final part I study equilibrium effort inequality, defined as $\max_i\{x_i^*\} - \min_i\{x_i^*\}$, and equilibrium payoff inequality, defined as $\max_i\{u_i(\mathbf{x}^*)\} - \min_i\{u_i(\mathbf{x}^*)\}$. By the earlier-mover advantage result discussed above, these expressions are equivalent to $x_1^* - x_n^*$ and $u_1(\mathbf{x}^*) - u_n(\mathbf{x}^*)$ respectively. It is easy to see that both expressions are minimized by the simultaneous contest since this is the only contest where efforts and payoffs are equal for all players.

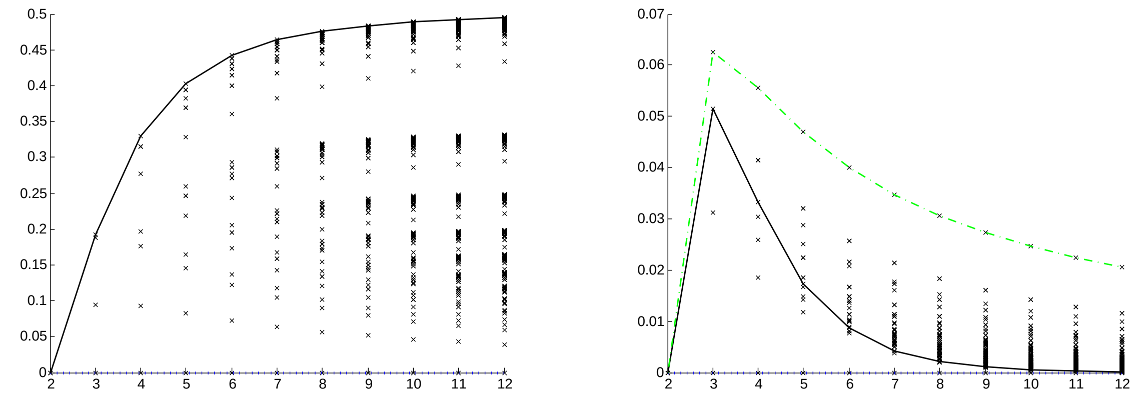
Maximization of effort inequality and payoff inequality leads to additional trade-offs. Consider the effort inequality first. As seen above, minimizing the lowest effort requires an informative contest. On the other hand, maximizing highest effort may involve pooling some of the later-movers. Therefore the effort inequality is maximized by a contest that is (weakly) more informative than the contest that maximizes the highest effort. Indeed,

[9] This comes from the independence of permutations result from Hinnosaar (2018), which shows that total equilibrium effort X^* is independent of permutations of vector $\mathbf{n} = (n_1,\ldots,n_T)$.

[10] Remark: when n is very large, $h(X^*) \to h(\overline{X}) = 0$ and therefore $u_i(\mathbf{x}^*) \to 0$ for all i, which means that technically all contests are approximately optimal. However, as figure 1b illustrates, in some contests this convergence is much faster than in others. By optimality I mean here the contest for which the value of the objective is maximized for large but finite values of n.

as the figure figure 2a shows, in Tullock contests with up to 12 players the first effect dominates and the optimal contest is sequential. Let us compare again Tullock contests $\mathbf{n} = (1, 2)$ and $\widehat{\mathbf{n}} = (1, 1, 1)$. The first contest leads to maximized highest effort $x_1^* = 0.375$, but also relatively high lowest effort $x_3^* = 0.1875$. The sequential contest $\widehat{\mathbf{n}}$ leads to slightly lower highest effort $\widehat{x}_1^* \approx 0.3591$, but the impact to the lowest effort is bigger, $\widehat{x}_3^* \approx 0.1667$. Therefore sequential contest maximizes the effort inequality.

The formal proposition proves these qualitative properties. It shows that the payoff-inequality-maximizing contest must have a single leader and it cannot be less informative than any contest that maximizes the highest effort. In the case when $h(X)$ is linear or n is large, the most informative maximizer of the highest effort is the sequential contest. Therefore, we can immediately conclude that the sequential contest must also be the unique maximizer of the equilibrium effort inequality, which confirms the result from figure 2a.



(a) Effort inequality

(b) Payoff inequality. Green dot-dashed line indicates the first-mover contest.

Figure 2: Effort inequality $\max_i\{x_i^*\} - \min_i\{x_i^*\}$ and payoff inequality $\max_i\{u_i(\mathbf{x}^*)\} - \min_i\{u_i(\mathbf{x}^*)\}$ in all normalized Tullock contests with $2 \leq n \leq 12$ players. The solid black line indicates sequential contests and simultaneous contests lie on the horizontal axis.

Finally, maximizing payoff inequality has an additional first-order effect: disclosures increase X^* and thus reduce all payoffs. This effect mechanically reduces payoff inequality as $u_1(\mathbf{x}^*) - u_1(\mathbf{x}^*) = (x_1^* - x_n^*)h(X^*)$. In the three-player examples above, $x_1^* - x_n^* = 0.1875 < \widehat{x}_1^* - \widehat{x}_3^* \approx 0.1925$. However, $h(X^*) \approx 0.3333 < \widehat{X}^* \approx 0.2679$. Therefore this additional effect is larger and $\mathbf{n} = (1, 2)$ is the three-player contest that maximizes payoff inequality. Figure 2b shows that in the case of Tullock contests, this effect is strong enough so that for all $n \leq 12$, the contest that maximizes payoff inequality is always the first-mover contest $\mathbf{n} = (1, n-1)$.

The proposition below shows that the contest that maximizes payoff inequality must have a single leader. It is more informative than any maximizer of the highest payoff and less informative than any maximizer of the effort inequality. When $h(X)$ function is linear or the contest is large, these conditions, unfortunately, are not very informative. All single-leader contests are more informative than the highest payoff maximizer $(1, n-1)$ and all contests are less informative than the effort inequality maximizer, which is the sequential contest. The proposition shows that we can in this case nevertheless uniquely determine that the optimal contest is the first-mover contest $\mathbf{n} = (1, n-1)$.[11]

Proposition 5 (Effort Inequality and Payoff Inequality)**.** *Both the equilibrium effort inequality* $\max_i\{x_i^*\} - \min_i\{x_i^*\}$ *and the equilibrium payoff inequality* $\max_i\{u_i(\mathbf{x}^*)\} - \min_i\{u_i(\mathbf{x}^*)\}$ *are minimized by the simultaneous contest.*

If contest $\mathbf{n}$ *maximizes the equilibrium effort inequality, then* $n_1 = 1$ *and* $\mathbf{n}$ *is not less informative than any maximizer of the highest effort. If* $h(X)$ *is linear or* n *is large enough, then* $\mathbf{n}$ *is the sequential contest.*

Any contest $\mathbf{n}$ *that maximizes the equilibrium payoff inequality must satisfy* $n_1 = 1$*. Moreover, it cannot be less informative than any maximizer of the highest payoff and it cannot be more informative than any maximizer of the effort inequality. If* $h(X)$ *is linear or* n *is large enough, then* $\mathbf{n}$ *is the first-mover contest.*

4 Conclusions

In this paper, I studied contest architecture with public disclosures. Additional disclosures induce discouragement effect that leads to higher efforts by earlier-movers, lower efforts by later-movers, and higher total effort. These effects have different implications to different objectives a contest designer may have. The optimal contests are summarized by table 1. Perhaps the most surprising finding of the paper is that for most objectives the optimal contest is one of the three standard contests the previous literature has been focused on—the simultaneous contest, the sequential contest, or the first-mover contest.

<table>
<tr><th colspan="2">Objective function</th><th>Minimizer</th><th>Maximizer</th></tr>
<tr><td>Total effort</td><td>$X^* = \sum_{i=1}^n x_i^*$</td><td>Simultaneous</td><td>Sequential</td></tr>
<tr><td>Total welfare</td><td>$\sum_{i=1}^n u_i(\mathbf{x}^*)$</td><td rowspan="3">Sequential</td><td rowspan="3">Simultaneous</td></tr>
<tr><td>Lowest effort</td><td>$\min_i\{x_i^*\}$</td></tr>
<tr><td>Lowest payoff</td><td>$\min_i\{u_i(\mathbf{x}^*)\}$</td></tr>
<tr><td>Highest effort</td><td>$\max_i\{x_i^*\}$</td><td>Simultaneous</td><td>Single-leader†</td></tr>
<tr><td>Highest payoff</td><td>$\max_i\{u_i(\mathbf{x}^*)\}$</td><td>$(2, 1, \ldots, 1)^\dagger$</td><td>First-mover†</td></tr>
<tr><td>Effort inequality</td><td>$\max_i\{x_i^*\} - \min_i\{x_i^*\}$</td><td rowspan="2">Simultaneous</td><td>Sequential†</td></tr>
<tr><td>Payoff inequality</td><td>$\max_i\{u_i(\mathbf{x}^*)\} - \min_i\{u_i(\mathbf{x}^*)\}$</td><td>First-mover†</td></tr>
</table>

Table 1: Summary of the optimal contests for the $8 \times 2 = 16$ objectives discussed in the paper. Contests marked with † are optimal at least for (1) Tullock contests with $n \leq 12$, (2) contests with linear $h(X)$ function for any n, and (3) large contests. The simultaneous contest is (n), the sequential contest is $(1, 1, \ldots, 1)$, the first-mover contest is $(1, n-1)$, and single-leader contest are $(1, n_2, \ldots, n_T)$ for all $n_2, \ldots, n_T$ (including the sequential and the first-mover contests).

[11] Again, subject to a remark: when n becomes large then all payoffs converge to zero and therefore any contest approximately maximizes the payoff inequality. However as figure 2b clearly illustrates, $u_1(\mathbf{x}^*) - u_n(\mathbf{x}^*)$ in the first-mover contest converges to zero much faster than other contests. The optimal contest here means that it maximizes the payoff inequality with finite but arbitrarily large n.

References

Acemoglu, D. and M. K. Jensen (2013): "Aggregate comparative statics," *Games and Economic Behavior*, 81, 27–49.

Bergemann, D. and S. Morris (2019): "Information Design: A Unified Perspective," *Journal of Economic Literature*, 57, 44–95.

Bimpikis, K., S. Ehsani, and M. Mostagir (2019): "Designing Dynamic Contests," *Operations Research*, 67, 339–356.

Che, Y.-K. and I. Gale (2003): "Optimal Design of Research Contests," *American Economic Review*, 93, 646–671.

Cournot, A.-A. (1838): *Recherches sur les principes mathématiques de la théorie des richesses par Augustin Cournot*, Chez L. Hachette.

Daughety, A. F. (1990): "Beneficial Concentration," *American Economic Review*, 80, 1231–1237.

Dixit, A. (1987): "Strategic Behavior in Contests," *American Economic Review*, 77, 891–898.

Doval, L. and J. Ely (2019): "Sequential information design," *manuscript.*

Ely, J. C. and M. Szydlowski (2019): "Moving the Goalposts," *Journal of Political Economy*, forthcoming.

Glazer, A. and R. Hassin (1988): "Optimal contests," *Economic Inquiry*, 26, 133–143.

——— (2000): "Sequential Rent Seeking," *Public Choice*, 102, 219–228.

Hinnosaar, T. (2018): "Optimal Sequential Contests," *manuscript.*

——— (2019): "Stackelberg Independence," *manuscript.*

Jensen, M. K. (2010): "Aggregative games and best-reply potentials," *Economic Theory*, 43, 45–66.

Kahana, N. and D. Klunover (2018): "Sequential lottery contests with multiple participants," *Economics Letters*, 163, 126–129.

Kamenica, E. and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

Konrad, K. A. (2009): *Strategy and Dynamics in Contests*, London School of Economics Perspectives in Economic Analysis, Oxford University Press, 1 ed., published: Paperback.

Martimort, D. and L. Stole (2012): "Representing equilibrium aggregates in aggregate games with applications to common agency," *Games and Economic Behavior*, 76, 753–772.

Moldovanu, B. and A. Sela (2001): "The Optimal Allocation of Prizes in Contests," *American Economic Review*, 91, 542–558.

——— (2006): "Contest architecture," *Journal of Economic Theory*, 126, 70–96.

Olszewski, W. and R. Siegel (2016): "Large Contests," *Econometrica*, 84, 835–854.

Robson, A. J. (1990): "Stackelberg and Marshall," *American Economic Review*, 80, 69–82.

Taylor, C. R. (1995): "Digging for Golden Carrots: An Analysis of Research Tournaments," *American Economic Review*, 85, 872–890.

Tullock, G. (1967): "The welfare costs of tariffs, monopolies, and theft," *Economic Inquiry*, 5, 224–232.

——— (1974): *The social dilemma: The economics of war and revolution*, University publications.

Vojnović, M. (2015): *Contest Theory: Incentive Mechanisms and Ranking Methods*, Cambridge: Cambridge University Press.

von Stackelberg, H. (1934): *Marktform und gleichgewicht*, J. springer.

Appendices for Online Publication

A Proofs

A.1 Useful Results

Notation: Following Hinnosaar (2018), it is useful to introduce additional notation. Let $g(X) = -\frac{h(X)}{h'(X)}$. With this, let us define $g_1(X), \dots, g_T(X)$ recursively as $g_1(X) = g(X)$ and $g_{k+1}(X) = -g_k'(X)g(X)$. For example, in the case of linear $h(X) = a(\overline{X} - X)$, $g(X) = \overline{X} - X = g_k(X)$ for all k. In the case of normalized Tullock payoffs $u_i(\mathbf{x}) = \frac{x_i}{X} - x_i$ and so $h(X) = \frac{1}{X} - 1$ and $g(X) = X(1 - X)$. Therefore $g_k(X)$ is a polynomial of degree $k + 1$.

For a contest $\mathbf{n} = (n_1, \dots, n_T)$ let us define its *measures of information*, denoted by $\mathbf{S}(\mathbf{n}) = (S_1(\mathbf{n}), \dots, S_T(\mathbf{n}))$, such as $S_k(\mathbf{n})$ is the sum of products of all possible k-combinations of the set $\{n_1, \dots, n_T\}$. For example, for contest $\mathbf{n} = (1, 2, 3)$, the first measure is $S_1(\mathbf{n}) = 1 + 2 + 3 = 6$. In fact, by construction $S_1(\mathbf{n}) = n$ for all contests. The second measure $S_2(\mathbf{n}) = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11$ captures the number of direct influences. The third measure $S_3(\mathbf{n}) = 1 \cdot 2 \cdot 3 = 6$ captures the number of indirect influences through one connection (for example, player 1 influences player 2, who in turn influences player 4). By construction, $S_k(\mathbf{n}) = 0$ for all $k > T$. Therefore $\mathbf{S}(\mathbf{n}) = (6, 11, 6)$ in the example.

For each period t, let $\mathbf{n}^t$ denote the *subcontest* starting after period t, i.e. $\mathbf{n}^t = (n_{t+1}, \dots, n_T)$. Then $\mathbf{S}(\mathbf{n}^t)$ are denoted analogously to the original contest. For example, if $\mathbf{n} = (1, 2, 3)$, then $\mathbf{n}^1 = (2, 3)$ and $\mathbf{n}^2 = (3)$, therefore $\mathbf{S}(\mathbf{n}^1) = (5, 6)$ and $\mathbf{S}(\mathbf{n}^2) = (3)$.

Theorem 1 (Characterization (Hinnosaar, 2018))**.** *The total equilibrium effort X^* is the highest root of $f_0(X) = 0$, where*

$$f_0(X) = X - \sum_{k=1}^{T} S_k(\mathbf{n}) g_k(X). \tag{1}$$

The equilibrium effort of player i in period t is $x_i^* = g_1(X^*) + \sum_{k=1}^{T-t} S_k(\mathbf{n}^t) g_{k+1}(X^*)$.

Informativeness and equilibria: As Hinnosaar (2018) shows, this result has strong implications for equilibrium behavior. The total equilibrium effort X^* is increasing in the informativeness of the contest. Therefore it is minimized by the simultaneous contest (the least informative contest) and maximized by the sequential contest (the most informative contest).

Independence of permutations: permutations in $\mathbf{n}$ do not change the total equilibrium effort X^*. For example, the contest $\mathbf{n} = (1, 2, 3)$ has exactly the same total effort as the contest $\widehat{\mathbf{n}} = (1, 3, 2)$ because $\mathbf{S}(\widehat{\mathbf{n}}) = \mathbf{S}(\mathbf{n})$. Of course, individual efforts may be affected, as they also depend on subcontest. But note that permutations within the subcontest do not affect individual equilibrium effort either: in the examples here, $\mathbf{n}^1 = (2, 3)$ and $\widehat{\mathbf{n}}^1 = (3, 2)$ is a permutation. Therefore $\widehat{x}_1^* = x_1^*$.

Earlier-mover advantage: take two players, i from period t and j from a later period $s > t$. Then $x_i^* > x_j^*$ and $u_i(\mathbf{x}^*) > u_j(\mathbf{x}^*)$. As explained in the text, this means that highest effort is always $\max_i\{x_i^*\} = x_1^*$, lowest effort $\min_i\{x_i^*\} = x_n^*$, highest payoff $\max_i\{u_i(\mathbf{x}^*)\} = u_1(\mathbf{x}^*)$, and lowest playoff $\min_i\{u_i(\mathbf{x}^*)\} = u_n(\mathbf{x}^*)$.

Large contests: Moreover, Hinnosaar (2019) shows that in the limit when the number of players becomes large, the equilibrium behavior converges to specific functional form. Intuitively, as the number of players becomes large and the total equilibrium effort is increasing in the number of players, it is quite natural that the total equilibrium effort converges to its full dissipation limit $\overline{X}$. As the function $h(X)$ is smooth, we can approximate it with a linear function near $\overline{X}$ arbitrarily closely when the number of players is large. Therefore it is not surprising that the equilibrium behavior of individual players converges to equilibrium behavior of the same game but with linear $h(X)$ function.

Theorem 2 (Competitive Limits (Hinnosaar, 2019))**.** *Fix a sequence* $\mathbf{n} = (n_1, \ldots, n_T)$ *and let us increase* n_t *at a particular period* t*, while keeping other elements of the sequence* $\mathbf{n}$ *unchanged. Then* $\lim_{n_t \to \infty} X^* = \overline{X}$ *and for each player* i *arriving in period* s*,*

$$\lim_{n_t \to \infty} x_i^* = \begin{cases} 0 & \forall s \geq t, \\ \frac{\overline{X}}{\prod_{k=1}^{s}(1+n_k)} & \forall s < t. \end{cases} \tag{2}$$

I use this result as follows. Note that n_t is not determined, as with different n the contest designer may want to use different disclosures. I use equation (2) as an approximation for the equilibrium behavior, i.e. each player $i \in \mathcal{I}_s$ chooses

$$x_i^* \approx \frac{\overline{X}}{\prod_{k=1}^{s}(1+n_k)}. \tag{3}$$

Note that this approximation also implies that

$$X^* = \sum_{i=1}^{n} x_i^* \approx \overline{X} - \frac{\overline{X}}{\prod_{k=1}^{T}(1+n_k)}. \tag{4}$$

Since $h(X)$ is a smooth function and $X^* \to \overline{X}$, we can approximate $h(X^*)$ linearly near $\overline{X}$, i.e.

$$h(X^*) \approx h(\overline{X}) + h'(\overline{X})(\overline{X} - X^*) = \frac{\alpha \overline{X}}{\prod_{k=1}^{T}(1+n_k)}, \text{ where } \alpha = -h'(\overline{X}) > 0. \tag{5}$$

A.2 Proof of Proposition 1 (Total Effort and Total Welfare)

Results for X^* follow from theorem 1. The total welfare is $W(\mathbf{x}^*) = X^* h(X^*)$, so

$$\frac{dW(\mathbf{x}^*)}{dX^*} = h(X^*) + X^* h'(X^*) \geq 0 \iff X^* \geq -\frac{h(X^*)}{h'(X^*)} = g(X^*). \tag{6}$$

Note that by theorem 1, each $x_i^* \geq g(X^*)$, therefore this condition is always satisfied. Each disclosure increases X^* and therefore decreases total welfare.

A.3 Proof of Proposition 2 (Lowest Effort and Lowest Payoff)

As argued in the text, $\min_i\{x_i^*\} = x_n^*$ and by theorem 1, $x_n^* = g(X^*)$. Since $g(X^*)$ is strictly decreasing in X^*, the lowest effort x_i^* is minimized when X^* is maximized, i.e. by sequential contest, and maximized when X^* is minimized, i.e. by simultaneous contest. Similarly, $\min_i\{u_i(\mathbf{x}^*)\} = u_n(\mathbf{x}^*) = x_n^* h(X^*) = g(X^*)h(X^*)$, where both g and h are decreasing in X^*, which leads to the same conclusion.

A.4 Proof of Proposition 3 (Highest Effort)

Let us first consider minimization of the highest equilibrium effort $\max_i\{x_i^*\} = x_1^*$. Let X^{sim} be the total equilibrium effort from simultaneous n-player contest (n). Then clearly the highest equilibrium effort $x_i^{sim} = \frac{X^{sim}}{n}$. Now take any non-simultaneous contest. As this contest is strictly more informative than the simultaneous contest, $X^* > X^{sim}$. Moreover, by the earlier-mover advantage result, the highest effort is strictly larger than the average, which proves the claim, since

$$x_1^* > \frac{X^*}{n} > \frac{X^{sim}}{n} = x_1^{sim}. \tag{7}$$

Suppose now that $\mathbf{n}$ maximizes the highest effort, which by theorem 1 is

$$\max_i\{x_i^*\} = x_1^* = g_1(X^*) + \sum_{k=1}^{T-1} S_k(\mathbf{n}^1) g_{k+1}(X^*). \tag{8}$$

I first claim that $n_1 < n_t$ for all t. If this is not the case, we can take a permutation of $\mathbf{n}$, which leaves X^* unchanged, but increases the information measures of the subcontest $\mathbf{n}^1$, therefore increasing the highest effort. Next, I claim that $n_1 = 1$. Suppose that this is not the case, i.e. $2 \le n_1$ But the previous observation, $n_2 \ge n_1 \ge 2$. Consider an alternative contest $\widehat{\mathbf{n}} = (n_1 - 1, n_2 + 1, n_3, \ldots, n_T)$. Then $\widehat{\mathbf{n}}^1 = (n_2 + 1, n_3, \ldots, n_T)$. Since $n_2 \ge n_1$, we have

$$(n_1 - 1)(n_2 + 1) = n_1 n_2 - n_2 + n_1 - 1 < n_1 n_2.$$

Therefore $S_2(\widehat{\mathbf{n}}) < S_2(\mathbf{n})$ and $\mathbf{S}(\widehat{\mathbf{n}}) < \mathbf{S}(\mathbf{n})$. Therefore $\widehat{X}^* < X^*$. This means that each $g_k(\widehat{X}^*) > g_k(X^*)$. Moreover, clearly $\mathbf{S}(\widehat{\mathbf{n}}^1) > \mathbf{S}(\mathbf{n}^1)$. Therefore $\widehat{x}_1^* > x_1^*$, which means that x_1^* was not the maximal highest effort.

Finally, when the $h(X)$ is linear or when number of players is large, then by the arguments above, the highest effort is (approximately) $x_1^* \approx \frac{X}{1+n_1}$. Clearly, this is maximized by setting $n_1 = 1$, i.e. a disclosure right after the first player. All other disclosures have no impact when $h(X)$ is linear and a negligible impact when n is large enough.

A.5 Proof of Proposition 4 (Highest Payoff)

Let $\mathbf{n}$ be a contest that minimizes the highest equilibrium payoff $\max_i\{u_i(\mathbf{x}^*)\} = u_1(\mathbf{x}^*) = x_1^* h(X^*)$. I claim that $n_1 \ge n_t$ for all t. If this is not the case, then there exists a permutation of $\mathbf{n}$ such that x_1^* is increased and X^* and thus $h(X^*)$ is unchanged. This would violate the optimality of $\mathbf{n}$. When n is large, then by the highest equilibrium payoff is approximately

$$u_1(\mathbf{x}^*) = x_1^* h(X^*) \approx \frac{\overline{X}}{1+n_1} \frac{\alpha \overline{X}}{\prod_{s=1}^T (1+n_s)}. \tag{9}$$

Minimizing this objective is equivalent to maximizing $(1+n_1)^2 \prod_{s=2}^T (1+n_s)$. We can do it in two steps. First, fix $n_1 \geq 1$ and choose optimal subcontest, which therefore needs to maximize $\prod_{s=2}^T (1+n_s)$ subject to constraint that $\sum_{s=1}^T n_s = n - n_1$. Splitting each $n_s \geq 2$ onto $n_s' > 0$ and $n_s'' > 0$ always increases the product as $(1+n_s')(1+n_s'') = 1 + n_s' + n_s'' + n_s' n_s'' > 1 + n_s' + n_s''$. Therefore the maximized product $\prod_{s=2}^T (1+n_s) = 2^{n-n_1}$ and so the maximization problem is

$$2^n \max_{1 \leq n_1 \leq n} (1+n_1)^2 2^{-n_1}. \tag{10}$$

Treating n_1 as a continuous variable and differentiating the objective gives

$$-2^{-n_1}(1+n_1)\left((1+n_1)\log 2 - 2\right) \leq 0 \quad \iff \quad n_1 \leq \frac{2}{\log 2} - 1 \approx 1.8854. \tag{11}$$

The objective is decreasing in n_1 for all $n_1 \geq 2$, so there are only two candidates for the maximizer: either $n_1 = 1$ or $n_1 = 2$. Direct comparison gives

$$(1+1)^2 2^{-1} = 2 < (1+2)^2 2^{-2} = \frac{9}{4}. \tag{12}$$

Therefore the optimal large contest is $\mathbf{n} = (2, 1, \ldots, 1)$.

Let $\mathbf{n}$ be a maximizer of the highest payoff $u_1(\mathbf{x}^*) = x_1^* h(X^*)$. The same arguments as in the proof if proposition 3 show that n_1 must be equal to 1 (first $n_1 \leq n_t$ for all t, otherwise can take a permutation, and then if $n_1 > 1$, splitting it makes the contest less homogeneous and thus decreases X^*, which increases total payoff while also increasing player 1's share in it). Now, let $\widehat{\mathbf{n}}$ be a contest that maximizes the highest effort. Let the corresponding effort profile be $\widehat{\boldsymbol{x}}^*$. Then by definition $x_1^* \leq \widehat{\boldsymbol{x}}^*$ and $u_1(\mathbf{x}^*) = x_1^* h(X^*) \geq \widehat{x}_1^* h(\widehat{X}^*) = u_1(\widehat{\boldsymbol{x}}^*)$, which cannot be satisfied unless $X^* \leq \widehat{X}^*$. Therefore $\mathbf{n}$ cannot be more informative than $\widehat{\mathbf{n}}$.

The final claim follows from the arguments above. With linear $h(X)$ or large n, one contest that maximizes the highest effort is $\widehat{\mathbf{n}} = (1, n-1)$. As the highest payoff maximizer must have $n_1 = 1$ and cannot be more informative than $\widehat{\mathbf{n}}$, it must coincide with $\widehat{\mathbf{n}}$.

A.6 Proof of Proposition 5 (Effort Inequality and Payoff Inequality)

As argued in the text, the simultaneous contest is the only contest where the efforts and payoffs of all players are equal, so it is the unique minimizer of both the equilibrium effort inequality $\max_i\{x_i^*\} - \min_i\{x_i^*\} = x_1^* - x_n^*$ as well as the equilibrium payoff inequality $\max_i\{u_i(\mathbf{x}^*)\} - \min_i\{u_i(\mathbf{x}^*)\} = u_1(\mathbf{x}^*) - u_n(\mathbf{x}^*)$.

By the same arguments as with the highest effort and with the highest payoff, we get that $n_1 = 1$ both with effort inequality and payoff inequality maximizers. Note that since x_1^* is independent of permutations of the subcontest $\mathbf{n}^1$ and $x_n^* = g(X^*)$ is independent of permutations in the whole contest, both inequalities are independent of permutations in $\mathbf{n}^1$.

Suppose that $\mathbf{n}$ is a effort inequality maximizer and let the corresponding equilibrium effort profile be $\mathbf{x}^*$. Take any highest effort maximizer $\widehat{\mathbf{n}}$ and let the corresponding effort profile be $\widehat{\boldsymbol{x}}$. Then by construction, $x_1^* \leq \widehat{x}_1^*$ and $x_1^* - x_n^* \geq \widehat{x}_1^* - \widehat{x}_n^*$, which implies that $g(X^*) = x_n^* \leq \widehat{x}_n^* = g(\widehat{X^*})$. As $g(X^*)$ is decreasing in X^* it implies $X^* \geq \widehat{X}^*$, which means that $\mathbf{n}$ cannot be less informative than $\widehat{\mathbf{n}}$. Finally, if $h(X)$ is linear or n is large, then any single-leader contest maximizes the highest effort. One such contest is the sequential contest, which is more informative than any other contest. Therefore it must be the unique maximizer of the effort inequality in these cases.

Now, let $\mathbf{n}$ be the maximizer of the equilibrium payoff inequality. As argued above, $n_1 = 1$. Take any maximizer of the highest payoff $\widehat{\mathbf{n}}$. Then $u_1(\mathbf{x}^*) - u_n(\mathbf{x}^*) \geq u_1(\widehat{\boldsymbol{x}}^*) - u_n(\widehat{\boldsymbol{x}}^*) \geq u_1(\mathbf{x}^*) - u_n(\widehat{\boldsymbol{x}}^*)$ and therefore $g(X^*)h(X^*) = u_n(\mathbf{x}^*) \leq u_n(\widehat{\boldsymbol{x}}^*) = g(\widehat{X^*})h(\widehat{X^*})$, which implies that $X^* \geq \widehat{X}^*$. This means that $\mathbf{n}$ cannot be less informative than $\widehat{\mathbf{n}}$. Next, suppose that $\widehat{\boldsymbol{x}}$ maximizes the effort inequality. Then $(x_1^* - x_n^*)h(X^*) \geq (\widehat{x}_1^* - \widehat{x}_n^*)h(\widehat{X}^*) \geq (x_1^* - x_n^*)h(\widehat{X}^*)$, which implies $h(X^*) \geq h(\widehat{X}^*)$ or equivalently $X^* \leq \widehat{X}^*$. Therefore $\mathbf{n}$ cannot be more informative than $\widehat{\mathbf{n}}$.

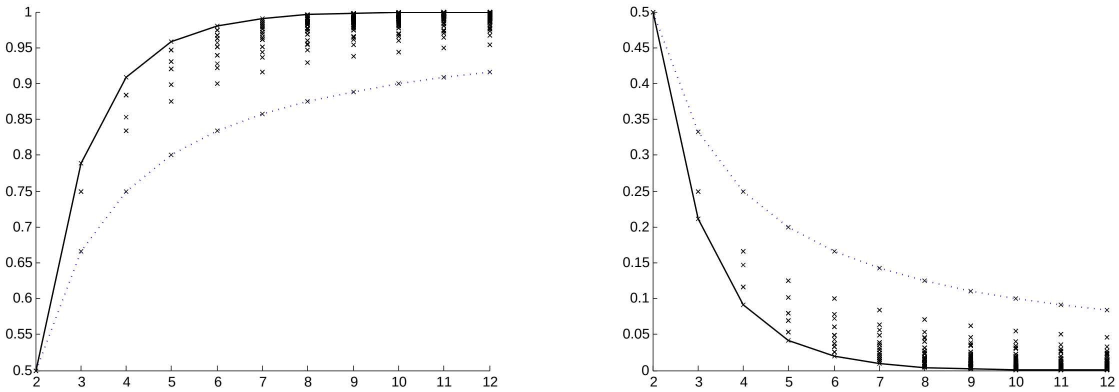
When $h(X)$ is linear or n is large, the results above do not tell us much, since none of the single-leader contests is less informative than $(1, n-1)$ (the maximizer of the highest payoff) and no contest is more informative than the sequential contest (the maximizer of the effort inequality). Therefore we have to proceed with direct proof. The equilibrium payoff inequality is (approximately)

$$
\begin{aligned}
u_1(\mathbf{x}^*) - u_n(\mathbf{x}^*) &\approx \frac{\overline{X}}{1+n_1} \frac{\alpha \overline{X}}{\prod_{s=1}^T (1+n_s)} - \frac{\overline{X}}{\prod_{s=1}^T (1+n_s)} \frac{\alpha \overline{X}}{\prod_{s=1}^T (1+n_s)} \\
&= \frac{\alpha \overline{X}^2}{(1+n_1)^2} \frac{1}{\prod_{s=2}^T (1+n_s)} \left[1 - \frac{1}{\prod_{s=2}^T (1+n_s)} \right].
\end{aligned} \tag{13}
$$

I already proved that $n_1 = 1$ for any payoffs, including linear. The remaining question is how to arrange the remaining $n-1$ players into subcontest $\mathbf{n}^1 = (n_1, \ldots, n_T)$. Let us denote $Y(\mathbf{n}^1) = \frac{1}{\prod_{s=2}^T (1+n_s)}$ for brevity. Then $Y(\mathbf{n}^1)$ is decreasing with each disclosure. Combining this with the fact that the optimal contest is not simultaneous, we get $Y(\mathbf{n}^1) \leq \frac{1}{1+n-n_1} \leq \frac{1}{2}$. Now, note that the expression $Y(\mathbf{n}^1)[1 - Y(\mathbf{n}^1)]$ is strictly increasing in $Y(\mathbf{n}^1)$ for all $Y(\mathbf{n}^1) < \frac{1}{2}$. Therefore maximization of payoff inequality requires that there are no disclosures after period 1, i.e. $Y(\mathbf{n}^1) = \frac{1}{1+n-n_1} = \frac{1}{n}$. This implies that the optimal contest is the first-mover contest $\mathbf{n} = (1, n-1)$.

B Additional Figures

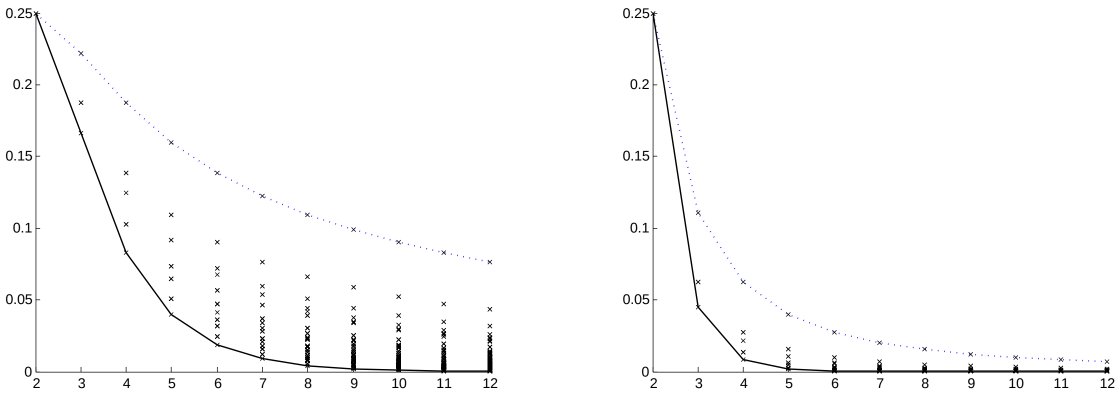
For completeness, I include here the figures describing the total equilibrium effort (figure 3a), total equilibrium welfare (figure 3b), lowest equilibrium effort (figure 4a), and lowest equilibrium payoff (figure 4b) in all Tullock contests with 2 to 12 players. The figures confirm fully general findings discussed in the text—in each of these eight possible cases the optimal contest is either the sequential contest or the simultaneous contest.



(a) Total effort

(b) Total welfare

Figure 3: Total equilibrium effort X^* and total equilibrium welfare $W(\mathbf{x}^*)$ in all normalized Tullock contests with $2 \leq n \leq 12$ players. The solid black line indicates sequential contests and dashed blue line simultaneous contests.



(a) Lowest effort

(b) Lowest payoff

Figure 4: Lowest effort $\min_i\{x_i^*\}$ and lowest payoff $\min_i\{u_i(\mathbf{x}^*)\}$ in all normalized Tullock contests with $2 \leq n \leq 12$ players. The solid black line indicates sequential contests and dashed blue line simultaneous contests.

C Minimizing or Maximizing the i-th Highest Effort and Payoff

In the case of linear $h(X)$ or large n, the problem is simple enough to consider other objectives. Consider for example i-th highest effort. By the earlier-mover advantage result, it is the effort of player $i \in \mathcal{I}_t$. If $h(X)$ is linear or n is large, it is (approximately)

$$x_i^* \approx \frac{\overline{X}}{\prod_{k=1}^t (1+n_k)}. \tag{14}$$

Remember the general trade-off: each disclosure reduces the marginal benefit of effort for all players and therefore pushes all efforts downwards, but it increases the efforts of earlier-movers whose effort is now made public. Therefore each disclosure before the arrival of player i only has the former effect and therefore reduces x_i^*. Therefore minimizing x_i^* require players before i to be arranged sequentially and maximizing requires them to be simultaneous (i.e. no disclosures prior to i). Disclosures after player i have the two opposite effects, so their impact depends on the relative magnitudes. However, in the case considered here (linear demand), they are exactly equal and therefore balance out. Therefore x_i^* is independent on the choice of disclosures after the period player i arrives.[12] This allows us to state the following simple proposition.

Proposition 6. *When $h(X)$ is linear or n is large*

1. *The contest that has $i-1$ first players arranged sequentially and the rest of the players arriving simultaneously, i.e. $\mathbf{n} = (1, \ldots, 1, n+1-i)$, minimizes the i-th largest effort.*

2. *Any contest that has i players in the first period, i.e. $\mathbf{n} = (i, n_2, \ldots, n_T)$, maximizes the i-th largest effort.*

When $i = n$ and $i = 1$ this result confirms propositions 2 and 3: lowest effort ($i = n$) is minimized by sequential and maximized by sequential contest, and highest effort ($i = 1$) is minimized by the simultaneous contest and maximized by any single-leader contest.

We can analogously study the i-th highest payoff $u_i(\mathbf{x}^*)$, which is (approximately)

$$u_i(\mathbf{x}^*) = x_i^* h(X^*) \approx \frac{\overline{X}}{\prod_{k=1}^t (1+n_k)} \frac{\alpha \overline{X}}{\prod_{k=1}^T (1+n_k)}. \tag{15}$$

Now there are three effects. First, all disclosures before the arrival of player i decrease both x_i^* and $h(X^*)$ and thus decrease $u_i(\mathbf{x}^*)$. Second, all disclosures after period t (when i arrives) leave x_i^* unaffected, but decrease $h(X^*)$, thus decreasing $u_i(\mathbf{x}^*)$. And third, the length of period t, i.e. delaying the first disclosure after the arrival of player i, has two opposite effects—it reduces x_i^* but increases $h(X^*)$. Combining these effects gives us the following proposition.

Proposition 7. *When $h(X)$ is linear or n is large*

[12] This the *Stackelberg Independence* property of a standard sequential homogeneous goods oligopoly, studied in detail in Hinnosaar (2019).

1. *The contest that minimizes the i-th highest payoff has the following structure: $i-1$ sequential players first, then two players simultaneously, and then the remaining $n-i-1$ players sequentially.*

2. *The contest that maximizes the i-th highest payoff is $\mathbf{n} = (i, n-i)$ when $i \leq \bar{i} = -\frac{1}{2} + \sqrt{\frac{5}{4} + n}$ and the simultaneous contest when $i \geq \bar{i}$.*

This confirms the findings from proposition 2 (the lowest payoff is minimized by sequential and maximized by simultaneous contest) and proposition 4 (the highest payoff is minimized by contest $(2, 1, \ldots, 1)$ and maximized by the first-mover contest).

Proof. Let us first consider minimization of the i-th highest payoff $u_i(\mathbf{x}^*)$. The arguments state that the players before i should be sequential as well as players after period t. The remaining question is n_t, i.e. how many player to pool with i. This gives a maximization problem

$$\min_{1 \leq n_t \leq n+1-i} \frac{1}{2^{i-1}(1+n_t)} \frac{1}{(1+n_t)2^{n-n_t}} \iff \max_{1 \leq n_t \leq n+1-i} (1+n_t)^2 2^{-n_t}. \tag{16}$$

Treating n_t as a continuous variable and differentiating the objective gives necessary condition for optimality

$$2^{-n_t}(1+n_t)\left[2 - (1+n_t)\log 2\right] = 0. \tag{17}$$

Solving this gives $n_t = \frac{2}{\log 2} - 1 \approx 1.8854$. It is straightforward to verify that the objective is concave for all $1 \leq n_t \leq n$, so the only two candidates for the optimum are $n_1 = 1$ and $n_1 = 2$. The latter is bigger than the former as $(1+2)^2 2^{-2} = \frac{9}{4} > (1+1)^2 2^{-1} = 2$. Therefore optimal $n_t = 2$. We get that the optimal contest is $\mathbf{n} = (\underbrace{1, \ldots, 1}_{i-1}, 2, \underbrace{1, \ldots, 1}_{n-i-1})$.

Now let us consider maximization of the i-th highest payoff $u_i(\mathbf{x}^*)$. As argued above, all players before i should be pooled with i, so that player i arrives in period 1. All players after period t should be pooled together as well, which would lead to a two-period contest $(n_1, n-n_1)$ with $i \leq n_1 \leq n$ (in case $n_1 = n$ it is in fact a simultaneous contest). Therefore the maximization problem is equivalent to

$$\max_{i \leq n_1 \leq n} \frac{1}{1+n_1} \frac{1}{(1+n_1)(1+n-n_1)} \iff \min_{i \leq n_1 \leq n} (1+n_1)^2(1+n-n_1). \tag{18}$$

This problem is concave in the interior, therefore one of the two corners is optimal, either $n_1 = i$, which gives $(1+i)^2(1+n-i)$ as the value of the objective, or $n_1 = n$ which gives $(1+n)^2$. The former is the minimizer if and only if

$$(1+i)^2(1+n-i) \leq (1+n)^2. \tag{19}$$

Equalizing left-hand-side with the right-hand-side gives and solving for i gives three roots n and $-\frac{1}{2} \pm \sqrt{\frac{5}{4} + n}$. Lowest root is always negative, and the second one $\bar{i} = -\frac{1}{2} + \sqrt{\frac{5}{4} + n}$ always strictly between 1 and n.[13] Moreover, at $i = 0$ the condition is clearly violated. Therefore, we get that for low values $i \leq \bar{i}$ the condition is violated and therefore the optimal contest is $\mathbf{n} = (i, n-i)$, whereas for high values $i \geq \bar{i}$ the condition is satisfied and therefore the optimal contest is simultaneous $\mathbf{n} = (n)$. □

[13]Since $1 < \bar{i} = -\frac{1}{2} + \sqrt{\frac{5}{4} + n} \iff 1 < n$ and $n > \bar{i} = -\frac{1}{2} + \sqrt{\frac{5}{4} + n} \iff n^2 > 1$.