

Price setting on a network*

Very preliminary and incomplete.

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Abstract

Most products are produced and sold by supply chains, where an interconnected network of producers and intermediaries choose prices to maximize their profits. In this paper, I study equilibrium price-setting on a network where players observe upstream prices. I derive unique equilibrium and study its properties. The characterization allows studying multiple-marginalization, market power, mergers, and acquisitions in a unified framework. The results emphasize the importance of considering the information flows in regulatory policies. For example, the impact of a merger or acquisition crucially depends on the impact on the underlying information network and may either decrease or increase the welfare loss. Moving production from monopolistic to a competitive supplier does not necessarily increase welfare if it reallocates market power through changing the information flows.

JEL: C72, L14, D43

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1 Introduction

Most products are produced and sold by supply chains, where an interconnected network of producers, suppliers, and intermediaries choose prices to maximize their own profits. Some firms in the chain compete with others offering similar products or services, whereas others enjoy some market power in their niche. For example, in book publishing industry, a publisher purchases content from authors, services from editors and marketing firms, and outsources printing to a printer, who in turn purchases paper, ink, and other supplies from outside companies. The publisher is typically also not selling books directly to consumers, but has contracts with distributors, who in turn deal with retail chains and individual retailers. Many of these companies do not have close alternatives and therefore enjoy some market power. From a typical \$26 book, the retail side gets about half, printing

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[‡]The latest version of the paper is available at toomas.hinnosaar.net/pricing_network.pdf.

and transport costs are about \$3.25, and the author is generally paid about 15%, which means that the markups to various parties play a major role.¹

The main question of this paper is how the price of the final good is forming on these networks? In my model, I allow each node in the network to be either a competitive firm (price-taker) or a monopolist. Each firm has a constant fixed marginal cost and sets a price. Specifying the information flows and commitment power is of crucial importance. I model this as a directed acyclic graph, where players observe the prices (or the total price) of all the firms upstream from them. The theoretical model is quite general and allows various alternative interpretations.

To characterize the equilibrium prices of the final good and intermediate good prices, I introduce novel approach, where instead of characterizing best-responses of downstream firms (which is generally not feasible), I characterize the optimal behavior of downstream firms by inverted best-response functions. These functions can be computed directly from the costs and structure of the subnetworks following each player. I show that under quite general assumptions on the demand function, the equilibrium is unique and can be characterized for complex networks.

The results allow me to study multiple-marginalization, market power, mergers and acquisitions in a unified framework. The results emphasize the importance of specifying the whole underlying information network. A classic problem in these situations is double-marginalization² which arises whenever multiple monopolists choose their prices without taking into account negative externalities on the other firms through the reduced demand. In markets with multiple-marginalization, both firms and consumers, typically, benefit from mergers, which allow firms to internalize the externalities. This intuition continues to hold in networks, but the results show that this intuition is incomplete. A merger inevitably has an impact on the information sharing network and if the post-merger network is more connected (has more information according to the measures I define) then the impact could be negative both for welfare and profits. In fact, it is possible that a merger that reduces the direct double-marginalization (reduces the number of decision makers) and also reduces costs is still undesirable for a regulator. Similarly, moving production from monopolistic to a competitive supplier does not necessarily increase welfare if it reallocates market power through changing the information flows.

The equilibrium markups are higher for upstream firms and for firms who are followed by highly connected subnetworks. My equilibrium characterization provides a natural measure of market power for each monopolist in the network—the information measure of its downstream subnetwork. The equilibrium markups and profits of the firms are ranked according to this measure.

The topics considered in this paper have been thoroughly studied in two-level models with upstream and downstream markets.³ Riordan (1998) showed that if a dominant upstream firm merges with a competitive downstream firm, it may hurt the social welfare even if it leads to cost advantages upstream. Similarly, Salinger (1988); Ordober, Saloner, and Salop (1990) showed that a vertical merger between upstream and downstream firm can lead to market foreclosure and therefore this effect may dominate the benefits of

¹New York Times “Math of Publishing Meets the E-Book”, by Motoko Rich (Feb. 28, 2010), <https://www.nytimes.com/2010/03/01/business/media/01ebooks.html>.

²Or multiple-marginalization, since there are often more than two firms.

³See Salinger (1989) for an interpretation of a such model.

integration. Lin (1988) showed that sometimes a dominant firm may prefer to disintegrate to gain market power. Farrell and Shapiro (1990) studied horizontal mergers in (Cournot) oligopoly and showed that generally mergers lead to price decreases only if there are significant cost reductions. In contrast to this stream of literature, I study multi-level networks. The advantage of this is that I can shed light to issues that do not arise in two-level model, whereas tractability requires some new assumptions regarding the demand function and market structures.

Methodologically the paper builds on ideas from Hinnosaar (2018), that introduced the idea of inverted best-response approach as well as the measures of information that turn out to be useful in this setting as well. However, there are significant differences due to assumptions about the information and the payoffs. Hinnosaar (2018) studied contests with identical players and public disclosures, whereas here we need to allow heterogeneous costs as well as competitive and monopolistic players. Moreover, public disclosures allow solving for the equilibrium backwards, whereas with network structure it is inevitable to analyze the branches simultaneously. On the other hand, here we are making somewhat stronger assumptions about the payoff functions as extending the general conditions from Hinnosaar (2018) to network structure would be very demanding.

The paper proceeds as follows. Section 2 introduces a model with networked monopolists. Section 3 uses two simple examples to illustrate the difficulties of finding the equilibria and discusses how the approach I take in this paper overcomes these difficulties. Section 4 provides the characterization result, discusses its implications, shows how it applies in the linear demand case, and illustrates its applications by a more complex example. Section 5 extends the analysis to networks where some nodes may be competitive. Section 6 discusses the implications to multiple-marginalization and merger policy and section 7 discusses the connection between network structure and market power. Finally, section 8 concludes. The proof of the characterization result is postponed to appendix A.

2 Model

A finite number of players⁴ $\mathcal{N} \in \{1, \dots, n\}$ participate in a pricing game that determines a final price P of a good that is sold to final consumers with demand function $D(P)$, which is a strictly decreasing function with some \bar{P} such that $D(\bar{P}) = 0$. Moreover, I assume that the demand function satisfies the following assumption: let $g(P) = -\frac{D(P)}{D'(P)}$, then g is $d(\Gamma)$ -times monotone, i.e. for all $k = 1, \dots, d(\Gamma)$, derivative $\frac{d^k g(P)}{dP^k}$ exists and $(-1)^k \frac{d^k g(P)}{dP^k} \geq 0$.

The players are organized as nodes on a network $\Gamma = (\mathcal{N}, \mathcal{G})$, such that the edges denote observability of prices set by upstream firms. Formally, if an edge $(i, j) \in \mathcal{G}$, then i observes the price set by j . I assume that \mathcal{G} is a *directed acyclic graph*, i.e. there are no cycles⁵ and a downstream firm observes all the prices that all his upstream connections see, i.e. if there is a path from i to j , then also $(i, j) \in \mathcal{G}$. I longest path in the network by $d(\Gamma)$. Graphically, I illustrate observations by arrows from downstream firm i to upstream

⁴In the model, I include only strategic players (monopolists). In section 5 I show how the model naturally extends to situations where some parts of the supply chain operate on competitive markets.

⁵There are no paths $(i, i_1), (i_1, i_2), \dots, (i_m, i) \in \mathcal{G}$. In particular, $(i, i) \notin \mathcal{G}$.

firm j when i observes p_j . See figure 1 for an illustration. For visual clarity, I do not draw the edges that are implied by assumptions.⁶

Player $i \in \mathcal{N}$ chooses a price $p_i \geq 0$, observing all the upstream prices, i.e. each p_j , such that $(i, j) \in \mathcal{G}$. If $(i, j) \notin \mathcal{G}$, then i does not observe p_j . Player i also takes into account that p_i is observable to all players j such that $(j, i) \in \mathcal{G}$ and adjust their behavior optimally.

Player i has marginal costs $c_i \geq 0$ and maximizes profit $\pi_i(\mathbf{p}) = (p_i - c_i)D(P)$, observing upstream prices and knowing how downstream firms will respond. The profile of prices is denoted by $\mathbf{p} = (p_1, \dots, p_n)$, the profile of costs by $\mathbf{c} = (c_1, \dots, c_n)$, and the profile of realized profits by $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$. The network $\Gamma = (\mathcal{N}, \mathcal{G})$ and costs \mathbf{c} are commonly known. Note: in the main part of this paper I assume that each player is a local monopolist. In section 5 I extend the analysis to cases, where some nodes may operate in competitive market.

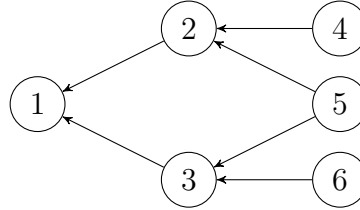


Figure 1: An example of a network with 6 players and longest path $d(\Gamma) = 2$. The edges that are implied by other edges (i.e. edges from 4, 5, and 6 to 1) are omitted for visual clarity.

3 Examples

In this subsection I first illustrate the two difficulties in characterizing the equilibria in price setting and then describe the approach I take to overcome these difficulties.

3.1 Difficulty 1: non-linear demand function

The first difficulty arises when working with non-linear demand functions and players who are best-responding to other players. To illustrate this issue, consider first a price setting with a simple two-player sequential price setting contest, with zero costs, and a demand function $D(P) = e - e^P$.⁷

Player 2 observes p_1 and maximizes

$$\max_{p_2 \geq 0} p_2 D(p_1 + p_2) \quad \Rightarrow \quad e - e^{p_1 + p_2} - p_2 e^{p_1 + p_2} = 0.$$

⁶For example, on figure 1, there are also edges from nodes 4, 5, and 6 to 1, but these edges are implied by the fact that there are indirect paths and adding the edges would complicate the figure.

⁷This function may arise for example in situations where buyers have unit demand and willingness to buy is some increasing function $V(\theta)$, where θ is distributed uniformly in $[0, 1]$. When $V(\theta) = \theta$, we get linear demand (that I will discuss next), but if V is non-linear then demand is non-linear. For example if $V(\theta) = \log(B + \theta)$, where $B \geq 1$ is added to avoid negative willingness to pay, then the implied demand function is $D(P) = 1 + B - e^P$. Taking $B = e - 1$ simplifies the expressions slightly.

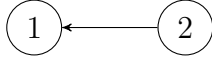


Figure 2: Sequential two-player network

Solving the optimality condition gives best-response function⁸ $p_2^*(p_1) = W(e^{2-p_1}) - 1$. Player 1's optimization problem is

$$\max_{p_1 \geq 0} p_1 D(p_1 + p_2^*(p_1))$$

and gives an optimality condition

$$e = e^{p_1 + W(e^{2-p_1}) - 1} \left[1 + \left(1 + \frac{-W(e^{2-p_1})}{1 + W(e^{2-p_1})} \right) p_1 \right].$$

Although the expression on the right-hand-side can be easily evaluated numerically for any p_1 , computing explicit expression for p_1^* is not feasible. Solving for equilibrium numerically is not difficult and gives $p_1^* \approx 0.5418$, $p_2^*(p_1^*) \approx 0.2417$, and thus $P^* = p_1^* + p_2^*(p_1^*) \approx 0.7835$ and $D(P^*) \approx 0.5291$.

However, when we would like to add one more player and study the three-player sequential pricing game, backward induction would require finding the best-response function for player 2 as a function of p_1 , which is not feasible.

To overcome this difficulty, I use the inverted best-response approach.⁹ Instead of deriving explicit expressions best-response functions $p_t^*(P_{t-1})$ (where P_{t-1} is the cumulative price prior to player t), it is more convenient to work with inverted best-responses $f_{t-1}(P)$, which tell for a given total effort (price), what the cumulative price P_{t-1} had to be before player t for the optimal choices of players t, \dots, n to lead to total price.

In particular, with two-player sequential game, trivially $f_2(P) = P$. Then the first-order condition of player 2 implies $f_1(P) = P - f_2'(P)(e^{1-P} - 1)$ and then the first-order condition of player 1 gives

$$f_0(P) = -e^{1-P} + P + 2 - e^{2-2P} \Rightarrow P^* \approx 0.7835,$$

i.e. the same solution as above, and in the three-player sequential contest we apply the same logic one more time and get,

$$f_0(P) = -e^{1-P} + P + 3 - 2e^{3-3P} \Rightarrow P^* \approx 0.8908.$$

The same approach can be used for computing equilibria for any n -player sequential game. Table 1 describes the numeric values for equilibrium prices for $n = 1, \dots, 9$.

⁸Where $W(z)$ denotes the Lambert's W function, defined as the solution to $W(z)e^{W(z)} = z$, because

$$\begin{aligned} e &= e^{p_1 + p_2} (1 + p_2) \\ 2 - p_1 &= 1 + p_2 + \log(1 + p_2) \\ e^{2-p_1} &= (1 + p_2)e^{1+p_2} \\ 1 + p_2 &= W(e^{2-p_1}). \end{aligned}$$

⁹Introduced in Hinnosaar (2018) for sequential contests with identical players and public disclosures.

n	P^*	p_1^*	p_n^*
1	0.5571	0.5571	0.5571
2	0.7835	0.5418	0.2417
3	0.8908	0.5313	0.1154
4	0.9443	0.5236	0.0572
5	0.9716	0.5179	0.0288
6	0.9855	0.5136	0.0146
7	0.9926	0.5104	0.0074
8	0.9963	0.5079	0.0037
9	0.9981	0.5060	0.0019

Table 1: Total price, first-mover's price p_1 and last-mover's price p_n in sequential n -player price setting game with demand function $D(P) = e - e^P$ and zero marginal costs

3.2 Difficulty 2: non-public disclosures

In addition to the difficulties with non-linear demand, the problem is non-trivial due to non-nested nature of information disclosure. Consider for example a four-player network illustrated by figure 3. Note that players 3 and 4 make their decisions independently and have different information available. Player 4 observes p_1 and p_2 , whereas player 3 observes only p_1 . In equilibrium players expect equilibrium behavior from players they do not observe, but when we characterize the equilibrium behavior of player 2, we need to take into account that the choice of player 2 affects players 4 but not player 3 (who does not observe deviations by player 2). This leads to another difficulty, where when characterizing the optimal behavior of a player, we cannot do it sequentially. The optimal behavior of player three cannot solved separately from players 2 and 4 and vice versa.

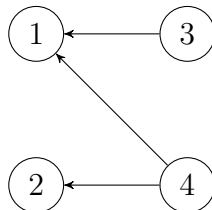


Figure 3: A simple non-nested network

To illustrate this more specifically, let us assume that the demand function is linear $D(P) = 1 - P$ and there are no costs. Note that the strategies of the players are respectively p_1^* , p_2^* , $p_3^*(p_1)$, and $p_4^*(p_1, p_2)$. Let us first consider the problem of player 4, who observes p_1 and p_2 and expects equilibrium behavior from player 3. Therefore player 4 solves

$$\max_{p_4 \geq 0} p_4 [1 - p_1 - p_2 - p_3^*(p_1) - p_4] \Rightarrow p_4^*(p_1, p_2) = \frac{1}{2} [1 - p_1 - p_2 - p_3^*(p_1)].$$

Player 3 solves a similar problem, but does not observe p_2 and thus expects p_4 to be $p_4(p_1, p_2^*)$, so

$$\max_{p_3 \geq 0} p_3 [1 - p_1 - p_2^* - p_3 - p_4^*(p_1, p_2^*)] \Rightarrow p_3^*(p_1) = \frac{1}{2} [1 - p_1 - p_2^* - p_4^*(p_1, p_2^*(p_1))].$$

To compute the best-response functions explicitly (i.e. independently of each other), we first need to solve the equation system that we get by inserting p_2^* to the optimality condition of player 4. This gives us

$$p_3^*(p_1) = p_4^*(p_1, p_2^*) = \frac{1}{3} [1 - p_1 - p_2^*].$$

With this we can compute

$$p_4^*(p_1, p_2) = \frac{1}{2} [1 - p_1 - p_2 - p_3^*(p_1)] = \frac{1}{3} [1 - p_1] + \frac{1}{6} p_2^* - \frac{1}{2} p_2.$$

Now, we can turn to optimization problem of player 2, who expects player 1 to choose equilibrium price p_1^* and thus

$$\max_{p_2 \geq 0} p_2 [1 - p_1^* - p_2 - p_3^*(p_1^*) - p_4^*(p_1^*, p_2)].$$

Taking the first-order condition and evaluating it at $p_2 = p_2^*$ gives a condition

$$\frac{1}{6} [2 - 2p_1^* - 5p_2^*] = 0. \quad (1)$$

Finally, player 1 solves a similar problem, taking p_2^* as fixed, i.e.

$$\max_{p_1 \geq 0} p_1 [1 - p_1 - p_2^* - p_3^*(p_1) - p_4^*(p_1, p_2^*)].$$

Again, taking the first-order condition and evaluating it at $p_1 = p_1^*$ gives

$$\frac{1}{3} [1 - 2p_1^* - p_2^*] = 0. \quad (2)$$

Solving the equation system equations (1) and (2) gives us $p_1^* = \frac{3}{8}$, $p_2^* = \frac{1}{4}$. Inserting these values to the best-response functions derived above¹⁰, we get that prices are $p_3^*(p_1^*) = p_4^*(p_1^*, p_2^*) = \frac{1}{8}$. Therefore $P^* = \frac{7}{8}$.

As the example illustrates, finding the equilibrium strategies requires a combination of equation systems in parallel with finding the best-response functions. Therefore each additional link can create a new layer of complexity.

I will show that $f_0(P)$, i.e. the function whose highest root determines the equilibrium, can be computed as

$$f_0(P) = P - \sum_{k=1}^{\infty} S_k(\Gamma) g_k(P),$$

where $g_1(P) = g(P) = -\frac{D(P)}{D'(P)}$ and $g_{k+1}(P) = -g'_k(P)g(P)$, so in the linear demand case $g_k(P) = 1 - P$ for all k , and $S_k(\Gamma)$ is the number of level k observations in game Γ . In the example here, we have four players, so $S_1(\Gamma) = 4$, and there are three direct observations of other player's choices, so $S_2(\Gamma) = 3$. There are no higher level observations. Therefore we get

$$f_0(P) = P - (1 - P)(4 + 3) = 8P - 7 = 0,$$

which confirms our finding that $P^* = \frac{7}{8}$.¹¹

¹⁰ Best-response functions are $p_3^*(p_1) = \frac{1}{4} - \frac{1}{3}p_1$, $p_4^*(p_1, p_2) = \frac{3}{8} - \frac{1}{3}p_1 - \frac{1}{2}p_2$.

¹¹The choices of individual players can be computed directly with the same approach as well: $p_3^* = p_4^* = g(P^*) = \frac{1}{8}$, $p_2^* = f'_2(P^*)g(P^*) = \frac{1}{4}$, $p_1^* = f'_1(P^*)g(P^*) = \frac{3}{8}$, where $f_1(P) = P - 2(1 - P)$ and $f_2 = P - (1 - P)$ are computed using the information measures for subnetworks following player 1 and player 2 respectively. I will discuss the details in the next subsection.

4 Characterization

4.1 Characterization result

The characterization result builds on the ideas introduced in the previous example. First, I use $f_i(P)$ to denote the *inverted best-response* function for player i . For any total price P , it describes what is the cumulative price P_i after player i 's choice that is consistent with P and the optimal behavior of players that observe i 's choice. Then the inverse function $f_i^{-1}(P_i)$ describes what will be the total price P , if after i 's choice the cumulative price is P_i . Note that P_i includes all prices that i cannot affect, including the ones that he does not observe. Then by construction $\frac{dP_i}{dp_i} = 1$, since the only price in the sum P_i that i 's choice affects is his own. I will show below that under the assumptions made here, each f_i is well-defined and invertible.

Similarly, $f_0(P)$ will capture the initial cumulative price necessary for reaching total price P . In this section, equilibrium requires that $f_0(P) = 0$ (in later section, when I allow price-takers, then it may be a positive constant).

Next, a *measure of information* of a network $\Gamma = (\mathcal{N}, \mathcal{G})$ is vector $\mathbf{S}(\Gamma)$, where $S_k(\Gamma)$ is the number of level- k observations in network Γ . In particular, $S_1(\Gamma)$ will always be the number of nodes. $S_2(\Gamma)$ the number of direct observations (i.e. the number of connected pairs of nodes), $S_3(\Gamma)$ is the number of two-step paths and so on. For example, in figure 3, there was four players, three direct observations and no paths of two or more steps, so $\mathbf{S}(\Gamma) = (4, 3)$. In the case of the network given by figure 1, $\mathbf{S}(\Gamma) = (6, 9, 4)$.

A *subnetwork following player i* , denoted by $\Gamma^i = (\mathcal{N}^i, \mathcal{G}^i) \subset \Gamma$ is such that $j \in \mathcal{N}^i$ if and only if there is a path $(j, j_1), (j_1, j_2), \dots, (j_m, i) \in \mathcal{G}$ (i.e. j observes i 's choice) and $(j, k) \in \mathcal{G}^i$ if and only if $j, k \in \mathcal{N}^i$ and $(j, i) \in \mathcal{N}$. For example, in figure 1, nobody observes the choices of players 4, 5, and 6, so their subnetworks are empty. As p_2 is observed by players 4 and 5, the subnetwork of player 2 consists of nodes 4 and 5, who do not observe each other's choices, thus there are no links. Therefore $\Gamma^2 = (\mathcal{N}^2, \mathcal{G}^2) = (\{4, 5\}, \emptyset)$. Figure 4 describe all three non-empty subnetworks of this network.

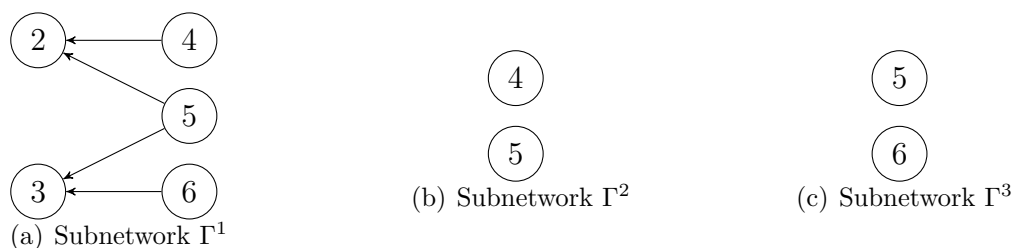


Figure 4: Non-empty subnetworks of the network in figure 1

Information measures of subnetworks are defined in the same way as for any network. For example, in the example from figure 1, we have that $\mathbf{S}(\Gamma^i) = (0)$ for $i \in \{4, 5, 6\}$, $\mathbf{S}(\Gamma^i) = (2)$ for $i \in \{2, 3\}$, and $\mathbf{S}(\Gamma^1) = (5, 4)$.

To shorten the notation, I denote the total marginal cost of subnetwork Γ^i by $C(\Gamma^i) = \sum_{j \in \mathcal{N}^i} c_j$ and the total marginal cost by $C = C(\Gamma) = \sum_{i=1}^n c_i$.

Finally, I define a sequence of functions $g_1(P), \dots, g_k(P)$ such that $g(P) = -\frac{D(P)}{D'(P)} = g_1(P)$, and $g_{k+1}(P) = -g'_k(P)g(P)$. Below I will show that each function $g_k(P)$ is contin-

uous and strictly increasing and discuss the interpretation of these functions.

We can now state the characterization result of this paper, which extends the ideas discussed in the previous section to arbitrary (directed acyclic) networks, players with arbitrary marginal costs, and any (m -times monotone) demand functions. As described in the previous section, it uses *inverted best-responses* to characterize the equilibrium. Formally, inverted best-response function of player i is going to be

$$f_i(P) = P - C(\Gamma^i) - \sum_{k=1}^{\infty} S_k(\Gamma^i)g_k(P). \quad (3)$$

The inverted best-response function captures the joint optimal behavior of all players affected by p_i , i.e. in subnetwork Γ^i . The interpretation is simple. Suppose that the sum of all prices that player i does not affect is P_{-i} . This is the sum of all the prices player i sees as well as the ones that she does not see and that are unaffected by his choice. Then after his choice, the cumulative price is $P_i = P_{-i} + p_i$ and the players after him behave optimally, which leads to some total price P . The function f_i captures the inverse relationship between P_i and P , i.e. assuming that players in subnetwork Γ^i behave optimally, i expects that P that follows P_i is $f_i^{-1}(P_i)$.

In particular, the inverted best-response function in the beginning of the pricing will be the one defined for the whole network and will determine the equilibrium price:

$$f_0(P) = P - C - \sum_{k=1}^{\infty} S_k(\Gamma)g_k(P). \quad (4)$$

Although it is not crucial to describe the individual best-response functions¹², for completeness I will also characterize them. For this, I need to re-invert the inverted best-response functions and need a bit of additional notation that will be only used for this purpose. To write player i 's choice as a function of values that he sees or infers, I use function $f_{-i}(P)$, which is the value of P_{-i} as a function of the final price P . Then player i 's choice is simply the difference of the cumulative after his choice minus the cumulative price before him, i.e. $P_i - P_{-i}$, both of which must be consistent with the a given P_{-i} and the fact that everyone starting from i behaves optimally, which gives

$$p_i^*(P_{-i}) = f_i(f_{-i}^{-1}(P_{-i})) - P_{-i}. \quad (5)$$

Theorem 1 (Characterization theorem). *There is unique equilibrium in price setting for network $\Gamma = (\mathcal{N}, \mathcal{G})$ with demand $D(P)$ and costs \mathbf{c} . The equilibrium price P^* is the unique solution to $f_0(P) = 0$, defined by equation (4).*

The individual equilibrium strategies are given by equation (5). In, particular, the equilibrium price of player i is $p_i^ = c_i + f_i'(P^*)g(P^*)$, where f_i is defined by equation (3).*

The proof is in the appendix A. The main idea of the proof is that when $f_i^{-1}(P_i)$ denotes the joint best-response of all players observing p_i by telling the player i what the total price P will if after the choice of p_i , the sum of all prices that i does not affect plus p_i is P_i . Then player i 's maximization problem is

$$\max_{p_i \geq 0} (p_i - c_i)D(f_i^{-1}(P_i)),$$

¹²The main objects of interest are the equilibrium price P^* and individual prices on the equilibrium path, i.e. p_i^* .

which implies optimality condition

$$D(f_i^{-1}(P_i)) + (p_i - c_i)D'(f_i^{-1}(P_i))\frac{df_i^{-1}(P_i)}{dP_i}\frac{dP_i}{dp_i} = 0$$

$$p_i - c_i = -\frac{D(P)}{D'(P)}f'_i(P) = f'_i(P)g(P). \quad (6)$$

Combining equations equation (6), taking into account the network structure, gives equations (3) to (5). In the proof I also verify that the equilibrium is unique and always in the interior, so the optimality condition (6) indeed the relevant one.

Note that equation (6) can be written as,

$$\frac{p_i - c_i}{p_i} = \frac{1}{\varepsilon(P)\frac{p_i}{P}}f'_i(P),$$

where $\varepsilon(P) = -\frac{D(P)}{PD'(P)} = \frac{P}{g(P)}$ is the elasticity of demand and the fraction $\frac{p_i - c_i}{p_i}$ is the Lerner index.

In a single player case this is the standard inverse elasticity rule. In the case there are more players, the rule is adjusted by two components. First, elasticity is multiplied by $\frac{p_i}{P}$, captures the mechanical price share of firm i 's decision in the final price. If this fraction is small, then the firm could increase the price more easily and thus choose a higher markup.

The final term $f'_i(P)$ captures the strategic effect. If firm i 's price is not observed by other players, this term is equal to 1. But if p_i is observed by the followers, then since prices are strategic substitutes, higher p_i discourages later players from choosing high prices, which means that the slope of the inverse function is typically lower than 1 and thus the slope of $f_i(P)$ is typically more than 1. Therefore, the more players observe p_i , the higher price i typically chooses.

4.2 The case of linear demand

Note that in the case of linear demand, we can without loss of generality normalize the demand function to $D(P) = 1 - P$. It is straightforward to check that then $g_k(P) = 1 - P$ for all k . Therefore equations (3) and (4) take the following forms

$$f_i(P) = (S_i + 1)P - C(\Gamma^i) - S_i,$$

$$f_0(P) = (S + 1)P - C - S,$$

where $S^i = \sum_{k=1}^{\infty} S_k(\Gamma^i)$ is the total number of all the information measures in subnetwork Γ^i and $S = \sum_{k=1}^{\infty} S_k(\Gamma)$ in the whole network. Therefore the equilibrium price P^* with linear demand is always

$$P^* = 1 - \frac{1 - C}{S + 1}. \quad (7)$$

Individual prices are therefore

$$p_i^* = c_i + f'_i(P^*)g(P^*) = c_i + (S_i + 1)[1 - P^*] = c_i + (1 - C)\frac{S_i + 1}{S + 1}. \quad (8)$$

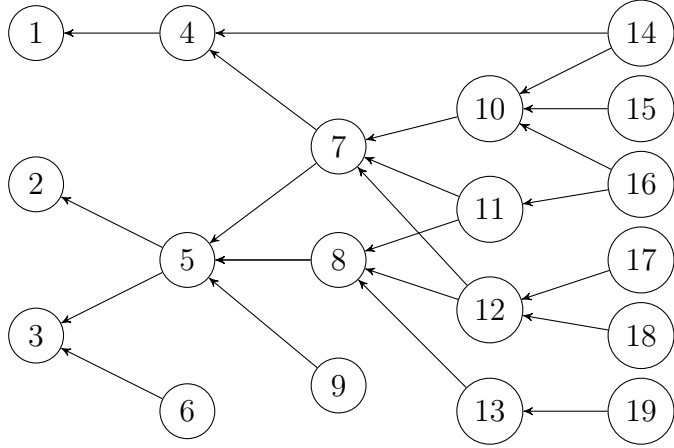


Figure 5: More complex network with 19 players. Note that the edges implied by other edges are omitted for visual clarity.

4.3 A more complex example

After some counting¹³ we get the information measures described by table 2

Network	S_1	S_2	S_3	S_4	S_5	$\sum_k S_k$
Γ^1	10	23	20	6		59
Γ^2	14	38	41	16		109
Γ^3	15	38	41	16		110
Γ^4	9	14	6			29
Γ^5	13	25	16			54
Γ^6						0
Γ^7	8	6				14
Γ^8	7	4				11
Γ^9						0
Γ^{10}	3					3
Γ^{11}	1					1
Γ^{12}	2					2
Γ^{13}	1					1
$\Gamma^{14}, \dots, \Gamma^{19}$						0
Γ	19	83	139	100	26	367

Table 2: Information measures on network described by figure 5.

Consider the network described by figure 5. Using theorem 1, we can now quite easily compute the equilibria for any given demand function (that satisfies the assumptions). Suppose first that the demand function is linear, i.e. $D(P) = 1 - P$ and costs are zero (i.e. as in section 3.2, but with more complex network). Then directly using equations (7) and (8) and table 2, we can conclude that $P^* = \frac{367}{368} \approx 0.9973$ and determine the individual price are $\frac{S_i+1}{368}$. For example, $p_1^* = \frac{60}{368} = \frac{15}{92} \approx 0.1630$, $p_2^* = \frac{110}{368} = \frac{55}{184} \approx 0.2989$, $p_3^* = \frac{111}{368} \approx 0.3016$ and so on, until the players with empty subnetworks having $p_i^* = \frac{1}{368} \approx 0.0027$.

¹³May be easier to do with a computer, on my website you will find a code for this.

The analysis is somewhat more complex with non-linear demand functions. Consider demand function $D(P) = e - e^P$ discussed in section 3.1. In this case

$$\begin{aligned} g_1(P) &= g(P) = -\frac{D(P)}{D'(P)} = e^{1-P} - 1, \\ g_2(P) &= -g'_1(P)g(P) = g(P)e^{1-P}, \\ g_3(P) &= -g'_2(P)g(P) = g(P)e^{1-P} (2e^{1-P} - 1), \\ g_4(P) &= -g'_3(P)g(P) = g(P)e^{1-P} (6e^{2-2P} - 6e^{1-P} + 1), \\ g_5(P) &= -g'_4(P)g(P) = g(P)e^{1-P} (24e^{3-3P} - 36e^{2-2P} + 14e^{1-P} - 1). \end{aligned}$$

Therefore taking $\mathbf{S}(\Gamma)$ from table 2 and computing $f_0(P)$ gives us an equilibrium condition

$$\begin{aligned} 0 &= f_0(P) = P - 19g_1(P) - 83g_2(P) - 139g_3(P) - 100g_4(P) - 26g_5(P) \iff \\ P &= 624e^{5-5P} - 960e^{4-4P} + 378e^{3-3P} - 24e^{2-2P} + e^{1-P} - 19 \end{aligned}$$

Solving the equation numerically gives us $P^* \approx 0.9973$. It is also relatively straightforward to determine the individual prices. In the case of players with empty subnetworks¹⁴ we get simply $p_i^* = g(P^*) \approx 0.0027$. For other players we need to compute $f'_i(P^*)$ first. For example, for players 1, 2, and 3 we get respectively

$$\begin{aligned} f'_1(P^*) &= 1 - 10g'_1(P^*) - 23g'_2(P^*) - 20g'_3(P^*) - 6g'_4(P^*) \approx 60.8342 \Rightarrow p_1^* \approx 0.1634, \\ f'_2(P^*) &= 1 - 14g'_1(P^*) - 38g'_2(P^*) - 41g'_3(P^*) - 16g'_4(P^*) \approx 111.7693 \Rightarrow p_2^* \approx 0.3002, \\ f'_3(P^*) &= 1 - 15g'_1(P^*) - 38g'_2(P^*) - 41g'_3(P^*) - 16g'_4(P^*) \approx 112.7720 \Rightarrow p_3^* \approx 0.3029. \end{aligned}$$

Note that the prices are similar, although not the same, as in the linear case. This is a general property. Functional form clearly affects the equilibria, but as the number of players increases a linear approximation near $P = 1$ becomes more and more accurate. Therefore in a network with large number of players it suffices to study the equilibria in the case of linear demand function, which is further simplified by equations (7) and (8).

5 Competitive firms

Often some parts of the supply chain are offered by firms operating in competitive markets. For example, it could be that the raw materials can be purchased from a competitive market at marginal cost. Then this could be included to the model without any alterations, simply by including these costs into the appropriate c_i values. However, sometimes it may be that there is no natural player whose cost is affected by such suppliers. In this case, we can simply create another cost term, say c_0 , which captures the price of all components produced by competitive firms. Then $C = \sum_{i=0}^n c_i$ and the rest of the model as well as theorem 1 are unchanged.

The following corollary shows that every time a monopolistic firm is replaced by a firm operating in a competitive market, both the total welfare and the total profit increase. Of course, the impact to particular firm i is negative, it earned positive profit as a monopolist and gets zero profit in the latter case.

¹⁴Players 6, 9, and 14–19.

Corollary 1. Consider price setting games $(\hat{\Gamma}, \hat{\mathbf{c}})$ and (Γ, \mathbf{c}) , where node $\hat{\Gamma}$ does not have node i nor any of the edges to and from i , but is otherwise identical to Γ , and $\hat{c}_j = c_j$ for all $j \neq 0, i$ and $\hat{c}_0 = c_0 + c_i$. Then the corresponding equilibrium prices satisfy $\hat{P}^* < P^*$.

Proof. Defining \hat{f}_0 and f_0 as the respective functions, from equation (4) we get

$$\hat{f}_0(P) - f_0(P) = \sum_{k=1}^{\infty} [S_k(\Gamma) - S_k(\hat{\Gamma})] g_k(P) > 0,$$

because by construction $\mathbf{S}(\hat{\Gamma}) > \mathbf{S}(\Gamma)$. Therefore

$$\hat{f}_0(P^*) > f_0(P^*) = 0 = \hat{f}_0(\hat{P}^*),$$

and thus $P^* > \hat{P}^*$. □

There could also be some cases where the same production input can be achieved in multiple ways. For example, suppose that in the network figure 1, production of component 2 can be achieved also by another component 2', with cost $c_{2'}$ and such that its price would be observable by 5 and 6, but not 4. In this case, of course the profit maximizing firms would choose the path that is cheaper and this would lead to one of two possibilities. One possibility is that the cheaper of the two options is so much cheaper that even its monopoly pricing outcome would be cheap enough to deter the alternative supplier to offer the service, in which case we could simply ignore this option in the equilibrium characterization. Alternatively, it could be that the costs of the two options are more similar, in which case the branches would engage in Bertrand competition and the cheaper branch would remain in the market, but as a price-taker (selling at the marginal cost of the more expensive branch). Therefore there is no reason to expect that this would change the calculations either. To avoid considering multiple cases, I simply assume in the rest of the paper that every component is produced either by a monopolistic supplier or perfectly competitive supplier.

6 Multiple-marginalization and mergers

The characterization result allows to shed some new light on the classic double-marginalization problem and merger policy. The standard theory says that if a production input for a monopolist is produced by another monopolist and both monopolist choose their optimal prices, then the total price is bigger and therefore the total welfare loss larger than when the two firms would be merged and maximize profits as one monopolist. Moreover, this double-marginalization is also worse for the monopolists' joint profit. This two-player analysis is a special case of my model.

6.1 Benchmarks

To evaluate the impact of multiple marginalization on a price-setting, I first compute two benchmarks: social optimum and joint monopoly optimum (i.e. maximum total profit).

Social welfare and social optimum: total welfare when price is P is

$$W(P) = \underbrace{\int_P^\infty D(x)dx}_{=\text{Consumer surplus}} + \underbrace{\sum_{i \in \mathcal{N}} (p_i - c_i)D(P)}_{=(P-C)D(P)=\text{Total profit}} = \int_P^\infty D(x)dx + (P - C)D(P).$$

Therefore, $W'(P) = (P - C)D'(P)$. As $D'(P) < 0$, we have that the socially optimal quantity is $Q^{fb} = D(C)$ and it is implemented by price $P^{fb} = C$. Any $P > C$ will necessarily lead to welfare loss

$$DWL(P) = W(C) - W(P) = \int_C^P [D(P) - D(x)]dx.$$

Notice that the socially optimal outcome gives zero profit for all firms, so P^{fb} is never an equilibrium in price setting game.

Joint profit maximization: an alternative objective would be joint profit maximization. The total profit

$$\Pi = \sum_{i \in \mathcal{N}} (p_i - c_i)D(P) = (P - C)D(P).$$

The optimality condition is

$$D(P) + (P - C)D'(P) = 0 \iff P - C = -\frac{D(P)}{D'(P)} = g(P).$$

Equivalently, monopoly optimum P^m must be a root of $f_m(P) = P - C - g(P)$. Comparing this expression with equation (4), we see that whenever $n > 1$, $f_0(P) > f_m(P)$ and therefore $f_0(P^m) > 0$, which means that $P^* > P^m > P^{fb} = C$. This confirms that the multiple-marginalization problem: both the social planner and joint monopoly maximizer would prefer to decrease the prices.

6.2 Multiple-marginalization problem

The next proposition shows that more information strictly increases the equilibrium price, which means total price and therefore magnifies the multiple-marginalization problem.

Proposition 1. *The equilibrium price P^* is strictly increasing in vector $\mathbf{S}(\Gamma)$.*

Proof. Suppose that $\hat{\Gamma}$ and Γ are such that $\mathbf{S}(\hat{\Gamma}) > \mathbf{S}(\Gamma)$ ¹⁵ and let \hat{P}^* and P^* be the corresponding equilibrium prices, which are roots of \hat{f}_0 and f_0 respectively. Then by equation (4) we have

$$\hat{f}_0(P) - f_0(P) = \sum_{k=1}^{\infty} [S_k(\Gamma) - S_k(\hat{\Gamma})] g_k(P).$$

Since $g_k(P) > 0$ for all $P < \bar{P}$ we have that $\hat{f}_0(P) < f_0(P)$. Therefore

$$\hat{f}_0(P^*) < f_0(P^*) = 0 = \hat{f}_0(\hat{P}^*).$$

Since \hat{f}_0 is strictly increasing (proved in lemma 1), we get that $\hat{P}^* > P^*$. □

¹⁵Where the comparison means that $S_k(\hat{\Gamma}) \geq S_k(\Gamma)$ for all k and $S_k(\hat{\Gamma}) > S_k(\Gamma)$ for at least one k .

We can use proposition 1 to evaluate the magnitude of the multiple-marginalization problem under various assumptions about the network Γ .

Corollary 2. *In the following cases $\mathbf{S}(\hat{\Gamma}) > \mathbf{S}(\Gamma)$ and therefore $\hat{P}^* > P^*$, so that the multiple-marginalization problem is bigger in $\hat{\Gamma} = (\hat{\mathcal{N}}, \hat{\mathcal{G}})$ than $\Gamma = (\mathcal{N}, \mathcal{G})$*

1. *If $\hat{\mathcal{N}} \supsetneq \mathcal{N}$ and $\hat{\mathcal{G}} = \mathcal{G}$, i.e. the new network has more players and no more information.*
2. *If $\hat{\mathcal{G}} \supsetneq \mathcal{G}$ and $\hat{\mathcal{N}} = \mathcal{N}$, i.e. the new network has more information and the same set of players.*

6.3 Mergers

A potential solution to multiple-marginalization problem is merging firms. As the benchmark calculations above showed, this is indeed beneficial here: a joint monopoly would be preferable both for the firms (jointly) and for the social planner.

However, not all mergers are socially beneficial and a key determinant for the benefits is how the merger impacts the information flows, i.e. the network structure. The following example figure 6 shows that a merger could reduce all the profits as well as the social welfare.

Suppose that the demand function is linear, i.e. $D(P) = 1 - P$ and there are no cost. Pre-merger network (figure 6(a)) is such that two components necessary parts of a product are offered by two distinct supply chains. Using equations (7) and (8) we can compute equilibrium and get $P^* = \frac{14}{15}$, $D(P^*) = \frac{1}{15}$, and profits $\pi_1^* = \pi_2^* = \frac{4}{225}$, $\pi_3^* = \pi_4^* = \frac{2}{225}$, $\pi_5^* = \pi_6^* = \frac{1}{225}$.

Now, suppose that firms 3 and 4 merge. That is, they are choosing the price \hat{p}_{3+4} jointly. A crucial question now is how this affects the network structure of information flows? One natural possibility is that the new firm 3 + 4 observes everything that firms observed separately and their joint behavior is observed by everyone who observed one of the firms before. This gives us a new network $\hat{\Gamma}$ on figure 6(b). Computing the equilibrium in the new network gives $\hat{P}^* = \frac{17}{18}$, $D(\hat{P}^*) = \frac{1}{18}$, so that the consumer surplus and social welfare have decreased. Moreover, $\hat{\pi}_1^* = \hat{\pi}_2^* = \frac{1}{54}$, $\hat{\pi}_5^* = \hat{\pi}_6^* = \frac{1}{324}$, so all the unaffected firms are worse off. Finally, $\hat{\pi}_{3+4}^* = \frac{1}{108} < \frac{4}{225} = \pi_3^* + \pi_4^*$. Therefore even the firms who merged are worse off.

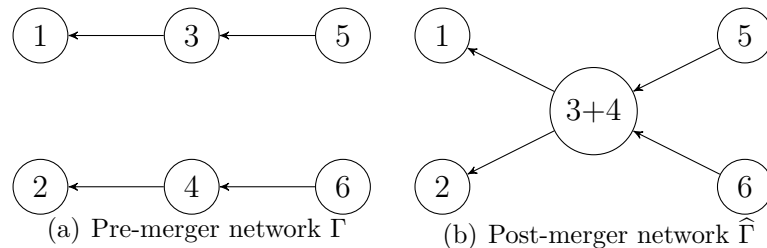


Figure 6: An example of a merger that is undesirable for all parties

Of course, this is somewhat extreme example and firms 3 and 4 would certainly not want to merge in this situation. However, it illustrates that the multiple-marginalization is

more nuanced problem than a standard intuition would suggest. In this example, merger is increases the multiple-marginalization problem, due to added information flows. Pre-merger measure of information was $\mathcal{S}(\Gamma) = (6, 6, 2)$ and post-merger is $\mathcal{S}(\hat{\Gamma}) = (5, 8, 4)$. Therefore $S_1(\hat{\Gamma}) < S_1(\Gamma)$, i.e. there are fewer decision makers in the new network, which reduces the multiple-marginalization problem by the standard argument. However, $S_2(\hat{\Gamma}) > S_2(\Gamma)$ and $S_3(\hat{\Gamma}) > S_3(\Gamma)$, i.e there is more information about other firms' prices observable. This increases stronger firms' incentive to raise markups, which in equilibrium leads to lower profits and welfare.

The next example with networks represented figure 7 shows a case where two firms prefer to merge and the merger reduces costs, but it is still socially undesirable. Suppose that the demand function is again linear $D(P) = 1 - P$. Suppose that initially the product is produced as a combination of processes from four separate monopolists (figure 7(a)), who all have equal marginal costs $c_i = \frac{1}{50}$, so that the total marginal cost is $C = \frac{4}{50}$. Then the equilibrium price is $P^* = \frac{102}{125} \approx 0.8160$ and individual profits $\pi_i^* = \frac{529}{15625} \approx 0.0339$.

Now, suppose that firms 1 and 2 merge in a way that leads to cost savings, i.e. then new marginal cost $\hat{c}_{1+2} = \frac{1}{50} < c_1 + c_2$. This means that the production is now more efficient and the total cost is $\hat{C} = \frac{3}{50} < C$. However, a key question is again how the merger affects the information flows. Suppose first that the merger does not change anything and the three new firms make their decisions in isolation (figure 7(b)). Then the new equilibrium price $\bar{P}^* = \frac{153}{200} = 0.765 < \hat{P}^*$ and profits $\bar{\pi}_i^* = \frac{2209}{40000} \approx 0.0552$ for all three firms. That is, the price is lower than before the merger, therefore both the social welfare and the joint profits are higher than before the merger. This is for two reasons: cost savings and reduction of double-marginalization. However, $\bar{\pi}_{1+2}^* < \pi_1^* + \pi_2^*$, so the firms 1 and 2 would not agree with the merger. This is so, because before the merger they got half of the whole markup and after the markup they only get a third.

Intuitively, there could be something missing from this analysis. By merging two parts of the supply chain, the new enlarged firm could have a stronger position in the market. One way to include this idea to the model is to assume that being bigger allows the new firm to pre-commit to a particular markup that the other firms must take into account. In particular, suppose that the merger gives the new larger firm more market power, so that its price \hat{p}_{1+2} is now observable for the other firms (figure 7(c)). Then we can apply equation (7) to get

$$\hat{f}_0(P) = P - \frac{3}{50} - 5(1 - P) \Rightarrow \hat{P}^* = \frac{247}{300} \approx 0.8233.$$

This is larger than the original market price P^* and therefore both the total welfare and the total profit are reduced. However, if we compute the individual profits, then

$$\hat{\pi}_{1+2}^* = \frac{2209}{30000} \approx 0.0736 > \pi_1^* + \pi_2^*, \quad \hat{\pi}_3^* = \hat{\pi}_4^* = \frac{2209}{90000} \approx 0.0245 < \pi_3^* = \pi_4^*.$$

Therefore for the firms 1 and 2 the merger is profitable. Of course, for the profits of firms 3 and 4 are reduced.

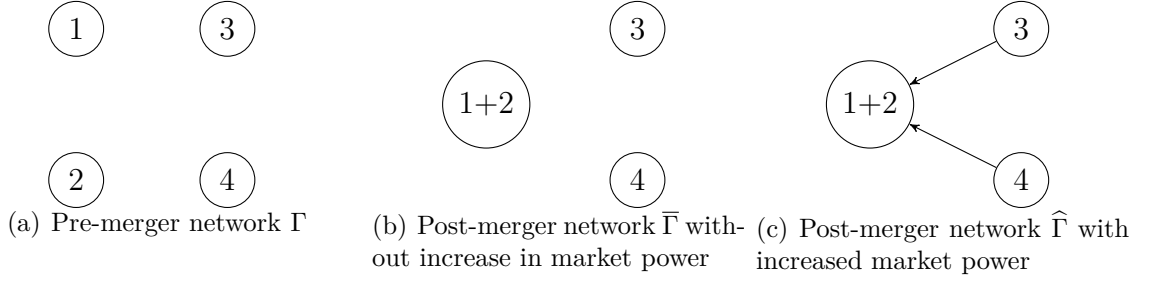


Figure 7: An example of a profitable merger is socially undesirable

7 Market power

The next question the characterization result allows to explore further is who are the stronger players in the network in terms of market power. More market power allows the firms to charge higher markups and achieve higher profits. Intuitively, upstream firms, i.e. the firms whose choices affect the following players, seem to have more impact and therefore have more power. As the following results show, this intuition is indeed correct. The results allow us also to

Proposition 2. *Take two players i, j in a price-setting game (Γ, \mathbf{c}) . If $\mathbf{S}(\Gamma^i) > \mathbf{S}(\Gamma^j)$, then the equilibrium markup $p_i^* - c_i > p_j^* - c_j$ and the equilibrium profit $\pi_i^* > \pi_j^*$.*

The opposite conclusion may not hold only because not all subnetworks can be ordered according to the informativeness $\mathbf{S}(\dots)$.

Proof. Since the game is fixed, it implies some equilibrium price P^* and demand $D(P^*)$. By theorem 1, prices of firms i and j satisfy

$$p_i^* - c_i = f'_i(P^*)g(P^*) = \left[1 - \sum_{k=1}^{\infty} S_k(\Gamma^i)g'_k(P^*) \right] g(P^*)$$

$$p_j^* - c_j = f'_j(P^*)g(P^*) = \left[1 - \sum_{k=1}^{\infty} S_k(\Gamma^j)g'_k(P^*) \right] g(P^*).$$

This implies that

$$(p_i^* - c_i) - (p_j^* - c_j) = \sum_{k=1}^{\infty} [S_k(\Gamma^i) - S_k(\Gamma^j)]g'_{k+1}(P^*) > 0$$

and therefore also

$$\pi_i^* - \pi_j^* = (p_i^* - c_i)D(P^*) - (p_j^* - c_j)D(P^*) > 0.$$

□

This result allows us to compare two firms i and j such that i 's price is observable to j , i.e. i is more upstream than j . In this case both i 's markup and profit will be higher than j 's.

Corollary 3. *If $j \in \mathcal{N}^i$, then $p_i^* - c_i > p_j^* - c_j$ and $\pi_i^* > \pi_j^*$.*

Follows directly from proposition 2, since $\mathcal{N}^j \subsetneq \mathcal{N}^i$ and $\mathcal{G}^j \subset \mathcal{G}^i$, so $S_k(\Gamma^i) \geq S_k(\Gamma^j)$ for all k and $S_1(\Gamma^i) > S_1(\Gamma^j)$.

The comparison is in fact even stronger, since it allows to compare the strength of the players who are making decisions without observing each others' prices. For example in the network described by figure 5, player 3 is the strongest since $\mathbf{S}(\Gamma^3) > \mathbf{S}(\Gamma^i)$ for all $i \neq 3$ (for calculations, see table 2). In particular, player 3 is stronger than player 2, because in addition to influencing every player that player 2 does, player 3 also influences player 6. Since the markups and profits are determined by subnetworks, it is not really important at which level the player is placed. For example, player 6 on the same network is influenced only by player 3, whereas player 10 is seemingly more downstream and is influenced by players 1, 2, 3, 4, 5, and 7. However, since player 6 does not influence anyone else, whereas player 10 influences three firms (14, 15, 16), player 10 is stronger than 6.

8 Conclusion

The paper characterizes equilibria in price setting on networks. Under quite general assumptions, the equilibrium exists, is unique, and can be computed using the tools developed here. I show that the multiple-marginalization problem gets magnified as either the number of players or information about the prices increases. There are situations where mergers can solve the problem and be beneficial for all firms as well as for the consumers. However, this depends crucially on how mergers affect the information networks. I also provided examples of merger scenarios, where a merger would be undesirable for firms and for the social planner. Upstream firms, who influence more of the following firms, have a larger market power and therefore achieve higher profits by committing to larger markups. The methodology extends also to supply chains where some parts of the production are offered by competitive firms. Replacing a monopolist by a competitive firm is beneficial both for the consumers and for the industry¹⁶

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¹⁶With the exception of the excluded monopolist, of course.

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A Proof of the characterization theorem

I will first prove a useful lemma about the properties of f_i functions defined by equation (3). First, note that by assumption $g(P)$ is $d(\Gamma)$ times monotone and decreasing. So, $g_1(P) = g(P)$ is also $d(\Gamma)$ times monotone and decreasing. If $g_k(P)$ is $d(\Gamma) + 1 - k$ times monotone and decreasing, then $-g'_k(P)$ is $d(\Gamma) + 1 - k - 1$ times monotone and decreasing. Therefore $g_{k+1}(P) = -g'_k(P)g(P)$ is $d(\Gamma) + 1 - (k + 1)$ times monotone and decreasing. Also, $g(\bar{P}) = 0$, therefore $g_k(\bar{P}) = 0$ for all k and

Lemma 1. *Function $f_i(P)$ has the following properties:*

1. $f_i(P)$ is strictly increasing and $d(\Gamma) + 1 - d(\Gamma^i)$ times monotone.
2. $f'_i(P)$ is $d(\Gamma) - d(\Gamma^i)$ times monotone and decreasing.
3. $f_i(P)$ has a unique root $\underline{P}_i \in [0, \bar{P}]$.

Proof. Proving each part separately:

1. Since $S_k(\Gamma^i) = 0$ for all $k > d(\Gamma^i)$ and $g_{d(\Gamma^i)}$ is $d(\Gamma) + 1 - d(\Gamma^i)$ times monotone and decreasing, equation (3) implies that $f_i(P)$ is a sum of functions, which are all at least $d(\Gamma) + 1 - d(\Gamma^i)$ times monotone and increasing. Moreover, since the first element P is strictly increasing, we have that $f_i(P)$ is strictly increasing.
2. As $f'_i(P)$ is a derivative of a $d(\Gamma) + 1 - d(\Gamma^i)$ times monotone increasing function, it is $d(\Gamma) - d(\Gamma^i)$ times monotone and decreasing.
3. By equation (3), $f_i(0) \leq 0$ (with equality if and only if Γ^i is empty) and $f_i(\bar{P}) = \bar{P} - C(\Gamma^i) > \bar{P} - C > 0$. Since $f_i(P)$ is strictly increasing and continuous, it follows that $f_i(P)$ has a unique root and it must be in $[0, \bar{P}]$.

□

Proof of theorem 1. Let us first consider any player with empty subnetwork Γ^i . Then trivially $f_i(P) = P$, since the only total price after i 's choice consistent with P is P itself, i.e. $f_i(P) = P$, which satisfies equation (3).

Now, take any player i and suppose that the best-responses of all the following players are characterized by inverted best-response function $f_i(P)$ given by equation (3). That is, whenever the sum of the prices player i does not affect and p_i is P_i , then the total price will reach $P = f_i^{-1}(P_i)$. Then player i 's maximization problem is

$$\max_{p_i \geq 0} (p_i - c_i) D(f_i^{-1}(P_i)).$$

The optimality condition is

$$D(f_i^{-i}(P_i)) + (p_i - c_i)D'(f_i^{-1}(P_i))\frac{df_i^{-1}(P_i)}{dP_i}\frac{dP_i}{dp_i} = 0.$$

Since in P_i player i affects only p_i , $\frac{dP_i}{dp_i} = 1$. Inserting $P = f_i^{-1}(P_i)$ and $\frac{df_i^{-1}(P_i)}{dP_i} = \frac{1}{f'_i(P)}$ gives

$$D(P) + (p_i - c_i)D'(P)\frac{1}{f'_i(P)} = 0 \iff p_i - c_i = -\frac{D(P)}{D'(P)}f'_i(P) = g(P)f'_i(P).$$

Note that this is a necessary condition for a locally optimal price p_i . Since $g(P)$ and $f'_i(P)$ are decreasing functions of P (by lemma 1) and $P = f_i^{-1}(P_i)$ is increasing function of p_i (again, by lemma 1), the expression on the right-hand-side is decreasing in p_i and therefore the equation has unique root. Moreover, as I will later verify, in equilibrium all players earn strictly positive profit. The corner solution of $p_i = 0$ would lead to non-positive profit. Therefore the local unique extremum of the optimization problem is also the unique maximum.

Now, note that $P_i = f_i(P)$ is the sum of p_i and all prices unaffected by i 's choice, denoted by P_{-i} (which includes all prices that i observes, as well as all prices that he does not see and infers in equilibrium). Assuming that i and all players in the subnetwork Γ^i behave optimally, P must satisfy

$$f_{-i}(P) = P_{-i} = P_i - p_i = f_i(P) - c_i - g(P)f'_i(P).$$

Note that since both $g(P)$ and $f'_i(P)$ are decreasing, and $f_i(P)$ is strictly increasing, the right-hand-side is strictly increasing, so the inverse function $f_{-i}^{-1}(P_{-i})$ is well-defined. Therefore for every P_{-i} , player i 's best-response function is indeed described by equation (5).

Next, I verify that if all players in Γ^i choose their best prices optimally as described above, then their behavior implies f_i given by equation (3). Indeed, each $j \in \mathcal{N}^i$ chooses p_j such that

$$p_j - c_j = g(P)f'_j(P) = g(P)\left[1 - \sum_{k=1}^{\infty} S_k(\Gamma^j)g'_k(P)\right] = g_1(P) + \sum_{k=2}^{\infty} S_{k-1}(\Gamma^j)g_k(P),$$

because $g_1(P) = g(P)$ and $g_{k+1}(P) = -g'_k(P)g(P)$. Therefore, since P_i plus the sum of p_j 's in \mathcal{N}^i is P , we have

$$f_i(P) = P - \sum_{j \in \mathcal{N}^i} p_j = P - C(\Gamma^i) - \underbrace{\sum_{j \in \mathcal{N}^i} g_1(P)}_{=S_1(\Gamma^i)g_1(P)} - \underbrace{\sum_{k=2}^{\infty} \sum_{j \in \mathcal{N}^i} S_{k-1}(\Gamma^j)g_k(P)}_{=S_k(\Gamma^i)},$$

which gives equation (3). The last equality comes from the facts that $S_1(\Gamma^i)$ is the number of players in Γ^i , which is equal to $\sum_{j \in \mathcal{N}^i} 1$, and $S_k(\Gamma^i)$ is the sum of all k -step paths in Γ^i , which can be decomposed into all components, where for each node in the subnetwork j we first look at the number of all $k-1$ -step paths in Γ^j (since there is an additional link from j to i , this adds one step).

We can compute $f_0(P)$ similarly as,

$$f_0(P) = P - \sum_{i \in \mathcal{N}} p_i = P - C - \underbrace{\sum_{i \in \mathcal{N}} g_1(P)}_{=S_1(\Gamma)g_1(P)} - \sum_{k=2}^{\infty} \underbrace{\sum_{i \in \mathcal{N}} S_{k-1}(\Gamma^i) g_k(P)}_{=S_k(\Gamma)},$$

which confirms equation (4).

Finally, the equilibrium must be consistent with the fact that prior to decisions of any players in Γ , the total price is 0, which means that $f_0(P) = 0$. By lemma 1, $f_0(P)$ is strictly increasing and has a unique root $P^* \in (0, \bar{P})$, which is then the unique candidate for an equilibrium.

Finally, note that since $P^* \in (0, \bar{P})$, at the equilibrium $D(P^*) > 0$ and $g(P^*) > 0$, and by lemma 1 also $f'_i(P^*) > 0$, we have that each firm sets $p_i^* - c_i > 0$ and therefore indeed gets strictly positive profit $(p_i^* - c_i)D(P^*) > 0$. \square