Robust Pricing with Refunds

Toomas Hinnosaar[*] Keiichi Kawai[†]

April 18, 2018

Abstract

We analyze a bilateral trade model where the seller has to make a take-it-or-leave-it offer to the buyer without knowing what the buyer has learned or will learn about the product fit. We show that a generous return policy reduces the significance of this type of uncertainty and helps the seller to regain market power. A simple mechanism that utilizes a generous return policy achieves the best guaranteed profit among all possible mechanisms. Our result provides a novel rationale for generous return policies.

1 Introduction

How should a seller price a new product? Should it set a high price to skim revenues from those who have high willingness-to-pay, or a low price to penetrate the market quickly and deeply? The answer to this question, among other things, crucially depends on how price sensitive the market is, i.e., the very information the seller does not possess. A lot of money may be left on the table if the seller simply uses existing products as their reference point to gauge the price elasticity of demand. The more novel the product may be, the more difficult it is to estimate the demand for the new product from existing products.

The companies that sell well-established products face similar uncertainty: They often are uncertain how well the consumers know about the product fit before purchase. For example, an online shoe store does not know whether or not the shopper has already tried a pair on at another store and knows what he/she wants. Similarly, a booking agent does not know whether a consumer, who is searching an airline ticket, already has a specific travel itinerary in mind, or comparing it with other options that are available to him/her.

In the environments mentioned above, the seller's uncertainty regarding buyer's information and learning severely limits its ability to extract rents from trade. Such uncertainty may encourage companies to take a broader view of the possibly complicated pricing strategies that go beyond a simple uniform pricing. What we show, however, is that a simple mechanism that uses a uniform price with a generous refund does the job even how costly the product return is to the seller.

We analyze a stylized bilateral trade model: a buyer's valuation of the product is either high or low, the exact value of which is unknown to everyone including the buyer. The

[*]Collegio Carlo Alberto, `toomas@hinnosaar.net`.

[†]UNSW, Sydney, `k.kawai@unsw.edu.au`.

seller and the buyer share a common prior about the buyer's valuation. The buyer observes a (costless) signal about her valuation. The seller neither knows the information structure (i.e. the distribution over signals) that the buyer observes, nor has a prior belief over possible information structures. The seller can offer a refund (and choose the amount) if the buyer decides to return the product after the purchase. The seller incurs a restocking cost if the product is returned. We consider three possible scenarios which differ in the timing at which the buyer receives the signal: (1) before the seller makes an offer; (2) after the seller made a (possibly randomized) offer, but before observing the realized offer; and (3) after observing the realized offer.[1]

For each scenario, our main results characterize the seller's best-guaranteed profit as a function of restocking costs. We also show that the seller maximizes its guaranteed profit by (exponentially) randomizing over the a refundable offer with a generous refund and (a continuum of) non-refundable offers. To understand the intuition, observe that the seller can diminish the significance of buyer's signal on her purchasing decision by offering a generous refund. The seller, however, needs to be wary of costs associated with product return. For each returned product, the seller incurs the restocking cost. The refundable offer that brings a balance to this trade-off is the one that comes with almost a full refund, but is attractive to the buyer if and only if her signal is sufficiently favorable. This way, the seller can ensure that the buyer who accepts the refundable offer always brings positive (ex-post) profit in expectation.

Loosely speaking, if the buyer's signal distribution is informative, then the buyer is likely to have a signal that persuades the buyer to accept the offer if and only if it is refundable. In contrast, if the buyer's signal distribution is not so informative, the buyer is likely to have a signal that persuade the buyer to accept if and only if the offer is non-refundable and the price is not sufficiently high. In this sense, the refundable offer works as a hedge against the distributions that are informative, and the randomization over non-refundable offers works as a hedge against the distributions that are not informative.[2] Also, even though the probability of doing so diminishes as the restocking cost increases, for any level of restocking cost, the seller makes a refundable offer (with a generous refund) with a positive probability. This is because the seller cannot exclude the possibility that the buyer's signal distribution is dispersed, for which the refundable offer is an effective hedge.

Our findings are closely related to the results in Roesler and Szentes (2017) and Du (2018). Roesler and Szentes (2017) identify the information structure that maximizes the buyer's welfare when the seller best responds to this information structure via uniform pricing. Du (2018) shows that the information structure found in Roesler and Szentes (2017) indeed minimizes the profit the seller can obtain; and the seller can obtain the best guaranteed profit by what he calls an exponential pricing regardless of the buyer's signal

[1]Each scenario, respectively, captures the environment where the seller knows that (1) the buyer cannot acquire any information, but the seller is uncertain what kind of information she has; (2) the buyer cannot acquire any information after observing the contract she faces, but may acquire additional information after learning the possible distribution over the offers she faces; and (3) nothing about the timing at which the buyer may acquire additional information.

[2]Notice that unlike the settings as in Inderst and Tirosh (2015) and Krahmer and Strausz (2015), where the seller knows the buyer's signal distribution, the seller is unable to choose a price-refund pair that brings a balance to this trade-off in expectation in the present paper. He instead needs to choose a price-refund pair (or randomization over the pairs) that guarantees him a certain profit independent of the buyer's signal distribution.

distribution.[3]

Unlike the present paper, their papers do not consider the possibility of product return. With the possibility of product return, the seller can potentially increase its profit through two channels. The seller can indirectly control the buyer learning by incentivizing the buyer to learn through purchase. The seller also can sequentially screen the buyer, first by the signal, and second by the realized valuation for the product.[4] Therefore, the selling mechanism that maximizes the seller's guaranteed profit may be convoluted. Nevertheless, we obtain the results parallel to Roesler and Szentes (2017) and Du (2018): a variant of the signal distribution identified in Roesler and Szentes (2017) is the worst possible buyer's signal distribution for the seller; and a variant of exponential pricing found in Du (2018) achieves the seller's best guaranteed profit agains any possible buyer's signal distributions.

Our findings offer a novel rationale behind a generous return policy. The literature has identified various reasons that companies may use return policies: e.g. a costly signals for product quality and product fit for the consumer (Grossman (1981); Moorthy and Srinivasan (1995); Inderst and Ottaviani (2013)); an insurance for risk-averse consumers (Che (1996)); a tool for price discrimination (Zhang (2013); Escobari and Jindapon (2014)).[5] Among those, the closest to our finding is Inderst and Tirosh (2015). In an environment where the seller knows the buyer's signal distribution, Inderst and Tirosh (2015) show that return policies work as "metering devices" or two-part tariffs (see Schmalense (1981)), where refunds make different consumers more similar and thus allow the firm to capture more of the surplus by raising prices.[6] Consequently, the seller sets the refund amount above the restocking cost. In contrast, our results show that the seller offers a generous refund (i.e., "almost" full refund) when the seller is uncertain of the buyer's signal distribution.

Lastly, we note that the model analyzed in the present paper can be interpreted as a game between the seller, who aims to maximize its profit by choosing a price-refund pair, and the adversarial nature, who aims to minimize the seller's profit by choosing the buyer's signal distribution. Thus, the game is akin to the Bayesian persuasion games with competing senders. We therefore fully utilize the concavification technique (Aumann et al. (1995); Kamenica and Gentzkow (2011)); and the linear structure of payoff functions in the competitive environment (Boleslavsky and Cotton (2018); Au and Kawai (2017a,b)) to derive the results.

2 Model

There are a (male) seller who can produce a product at no cost, and a (female) buyer whose valuation for the product is $v \in \{0, 1\}$. The buyer's valuation v follows a commonly known

[3]Libgober and Mu (2017) analyzes a robust dynamic pricing problem where the product is durable and buyers learn about their value for the product over time.

[4]The literature on sequential screening and dynamic mechanism design has identified why and how advance sales to still-uninformed consumers can help the seller. See e.g., Gale and Holmes (1992, 1993); Courty and Li (2000); Eso and Szentes (2007); Nocke et al. (2011); Gallego and Sahin (2010); Ely et al. (2017).

[5]Escobari and Jindapon (2014) also provide some empirical evidence on the use of refundable tickets by airlines. They show that fully refundable ticket is typically about 50% more expensive than a non-refundable ticket, but the difference disappears in the last week before the departure. These facts also fit well with our model predictions.

[6]Similar ideas have studied in other contexts, such as overbooking by airlines, e.g., Ely et al. (2017).

distribution such that $\pi = \Pr(v = 1)$. Neither the seller nor the buyer knows the realization of v. However, the buyer receives a signal about her valuation v prior to purchase. (Details will be explained momentarily.)

The seller's (pure) strategy is a contract (p, r) that specifies a sales price p together with a refund r. The seller may use a mixed strategy $\Delta\{(p, r)\}$. Based on the information she has, the buyer decides whether to buy after observing the contract $(p, r) \sim \Delta\{(p, r)\}$. We call the buyer's mixed strategy, i.e., a distribution over contracts, *a policy*; and the realized contract as *an offer*.

If the buyer purchases the product, then she learns the realized value of v. If $r = 0$, then the game ends. If $r > 0$, then the buyer decides whether or not to return the product. When the product is returned, the seller incurs a commonly known restocking cost $c > 0$. We sometimes use $\gamma \equiv \frac{c}{1+c} \in (0, 1)$ to denote the normalized restocking cost.

If the buyer keeps the product, or, when the realized offer is non-refundable, i.e., $r = 0$, then her payoff is $v - p$, and the seller's profit is p. If the realized offer is refundable, i.e., $r > 0$, and she returns, then her payoff is $r - p$, and the seller's profit is $p - r - c = p - r - \frac{\gamma}{1-\gamma}$, respectively. If she does not buy the product, then her payoff and the seller's profit are both zero.

We can represent a buyer's signal as a posterior distribution $q = \Pr[v = 1]$ over $v \in \{0, 1\}$ that is a random variable drawn from a distribution function $F \in \mathcal{F} \equiv \{F : \mathbb{E}_F[q] = \pi\}$.[7] For this reason, we use a distribution over posteriors F to represent the buyer's information structure, and call it *a signal distribution*. For the same token, by a *signal* q, we refer to the realization of the posterior distribution over v drawn from signal distribution F.

We are interested in the seller's *best profit guarantee* when he is uncertain of the buyer's signal distribution. More precisely, the buyer chooses a signal distribution F from a subset $\mathcal{F}_B$ of $\mathcal{F}$ that is unknown to the seller; and the seller neither has a prior distribution over $\mathcal{F}_B$ nor observes the buyer's choice of F. The timing at which the buyer chooses a signal distribution affects the seller's best profit guarantee.

Policy-independent signal: The buyer receives signal before the policy is announced, i.e. at $t = 1$ in Figure 1. The signal distribution $F \in \mathcal{F}$ is therefore independent of policy. This captures the environment where the seller knows that buyer cannot acquire any additional information, but she may have information that the seller is unaware of.

Policy-dependent signal: The buyer receives the signal after the policy is announced, i.e. at $t = 2$ in Figure 1. The signal distribution $F_{\Delta(p,r)} \in \mathcal{F}$ may depend on the policy $\Delta(p, r)$, but not the realization of (p, r). For example, the buyer may gather less information when she expects either very low or very high price (on average), or if the refund policy is very generous (with high probability).

Offer-dependent signal: The buyer receives the signal after the offer is realized, i.e. at $t = 3$ in Figure 1. The signal distribution $F_{(p,r)} \in \mathcal{F}$ may depend on the realized offer (p, r). In this case the buyer has full flexibility to adjust information gathering to particular price and refund offer she receives.

More formally, let $V((p, r) | F)$ be the seller's expected profit from offer (p, r) and the buyer's signal distribution after $t = 3$ is F. Similarly, $\mathbb{E}_{\Delta\{(p,r)\}} V((p, r) | F)$ represents the

[7] Notice that $F \in \mathcal{F}$ if and only if $\int_0^1 F(q)\, dq = 1 - \pi$.

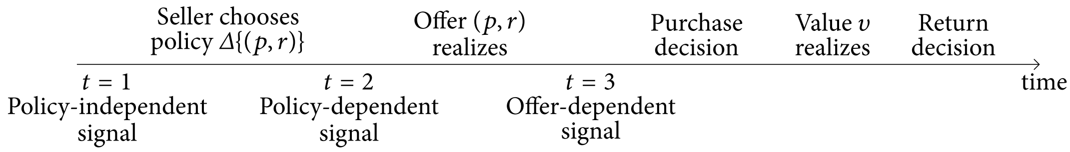


Figure 1: Timing

seller's expected profit from a policy $\Delta\{(p,r)\}$ when the buyer's signal distribution after $t = 3$ is F. The best profit guarantee the seller can obtain when the buyer chooses F at $t = 1, 2, 3$ are, respectively,

$$\begin{aligned} V_1^* &\equiv \sup_{(p,r)} \min_{F \in \mathcal{F}} V\left((p,r)\,|F\right), \\ V_2^* &\equiv \sup_{\Delta\{(p,r)\}} \min_{F_{\Delta\{(p,r)\}} \in \mathcal{F}} \mathbb{E}_{\Delta\{(p,r)\}} V\left((p,r)\,|F_{\Delta\{(p,r)\}}\right), \\ V_3^* &\equiv \sup_{(p,r)} \min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)\,|F_{(p,r)}\right). \end{aligned}$$

Obviously, $V_1^* \geq V_2^* \geq V_3^*$. We identify V_1^*, V_2^*, and V_3^*. Our results show that $V_1^* = V_2^*$ for all parameter values; and $V_2^* = V_3^*$ if and only if the (normalized) restocking cost γ is small enough. That is, if the restocking cost is sufficiently small for a given prior π, then the timing at which the buyer chooses her signal distribution does not affect the seller's best guaranteed profit. In contrast, when the restocking cost is sufficiently large, whether the buyer chooses a signal distribution after observing an offer do affect the seller's best guaranteed profit. That is, $V_1^* > V_3^*$. Nevertheless, our result that $V_1^* = V_2^*$ informs us that the seller can completely neutralize this negative effect by making a randomized offer and thereby by keeping the buyer in the dark about the actual offer she will face.

3 Analysis

3.1 Offer-Dependent Signal

We start with the case where the information structure may depend on the realized offer (p,r), i.e., the buyer may choose a signal distribution at $t = 3$. Our goal is to identify the best profit guarantee $V_3^* \equiv \sup_{(p,r)} \min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)\,|F_{(p,r)}\right)$. With a slight abuse of notation, let $V(q|\,(p,r))$ be the seller's expected profit when the offer is (p,r) and the buyer's signal is q. Then, $V\left((p,r)\,|F_{(p,r)}\right) = \mathbb{E}_{F_{(p,r)}}\left[V(q|\,(p,r))\right]$. Also for notational simplicity, we use $\underline{V}_3(p,r)$ to denote $\min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)\,|F_{(p,r)}\right)$, so that $V_3^* = \sup_{(p,r)} \underline{V}_3(p,r)$.

Suppose that the seller makes a non-refundable offer $(p,0)$. Then, the buyer with signal q buys if and only if $q \geq p$, i.e.,

$$V(q|\,(p,0)) = \begin{cases} 0 & q < p \\ p & q > p \end{cases}.$$

The seller's profit is minimized when the probability of signal being larger than p is minimized, i.e., when $F_{(p,0)}$ minimizes $1 - F_{(p,0)}(p)$ subject to $\mathbb{E}_{F_{(p,0)}}[q] = \pi$. More formally, we can derive $\underline{V}_3(p,0)$ by utilizing the concavification approach.[8] Let $\text{con}\left[-V\left(\cdot|\left(p,0\right)\right)\right](q)$ be the value of concave closure of $-V\left(\cdot|\left(p,0\right)\right)$ at q.[9] Then $\underline{V}_3(p,0) = -\text{con}\left[-V\left(\cdot|\left(p,0\right)\right)\right](\pi)$, which is represented by the red-dotted line in Figure 2.

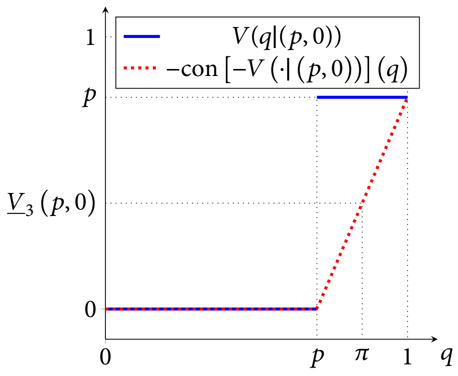


Figure 2: Profit of a non-refundable offer $(p,0)$

If $\pi > p$, then this occurs when the buyer's signal distribution induces two signals p (which results in no trade) and 1 (which results in trade) with probabilities $\frac{1-\pi}{1-p}$, and $\frac{\pi-p}{1-p}$, respectively. In contrast, if $\pi \leq p$, then this occurs when the buyer's signal does not disclose any additional information, i.e., induces signal π (which results in no trade) with probability one. While a higher p results in higher a profit margin should trade occur, it leads to a lower probability of trade. The seller brings a balance to this trade-off by offering $p = 1 - \sqrt{1-\pi}$. More formally,

$$\underline{V}_3(p,0) = -\text{con}\left[-V\left(\cdot|\left(p,0\right)\right)\right](\pi) = \begin{cases} 0 & p \geq \pi \\ p \times \frac{\pi-p}{1-p} & p < \pi \end{cases}$$

$$\leq \sup_p \underline{V}_3(p,0) = \sup_{p\in[0,\pi]} p\frac{\pi-p}{1-p} = \left(1-\sqrt{1-\pi}\right)^2. \tag{1}$$

Next, consider a refundable offer, i.e., (p,r) such that $r > 0$. Without loss of generality, we only consider the case where $p \geq r$.[10] Consider the buyer with signal q. Since her payoff from buying is $q \times 1 + (1-q) \times r - p$, she buys if only if her signal is above the marginal signal $\tilde{q}(p,r) \equiv \frac{p-r}{1-r}$; and returns with probability $1-q$ if she buys. The seller's profit from the buyer with signal q is thus

$$V\left(q|\left(p,r\right)\right) = \begin{cases} 0 & q < \tilde{q}(p,r) \\ \min\left\{0, v\left(q;p,r\right)\right\} & q = \tilde{q}(p,r) \\ v\left(q;p,r\right) & q > \tilde{q}(p,r) \end{cases}, \tag{2}$$

where $v(q;p,r) \equiv p - (1-q)(c+r) = p - (1-q)\left(\frac{\gamma}{1-\gamma} + r\right)$.

[8]See Aumann et al. (1995) and Kamenica and Gentzkow (2011).

[9]The concave closure of function G is defined by $\text{con}(G)(q) = \sup\{g|(q,g) \in \text{co}(G)\}$, where $\text{co}(G)$ is the convex hull of the graph of G.

[10]Notice that if $p < r$, then buyer buys irrespective of the value of signal. Therefore, $\underline{V}_3(p,r) < \underline{V}_3(p,r-\varepsilon)$ for a sufficiently small $\varepsilon > 0$.

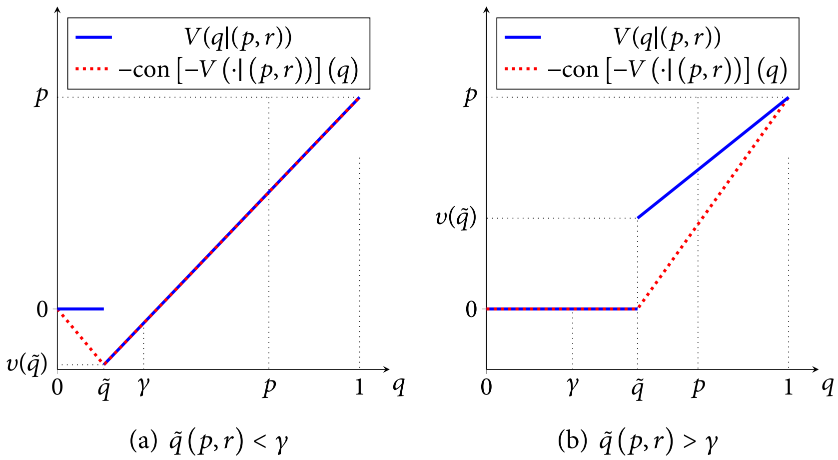


Figure 3: Profit from refundable offer (p, r)

Observe that if the marginal signal $\tilde{q}(p, r)$ is sufficiently low, i.e., if the refund is sufficiently generous, then the buyer buys even when her signal q is sufficiently low. The buyer with a low signal is likely to return the product, and thereby is likely to bring the seller an (ex-post) negative profit. More specifically, as captured by the increasing blue-lines in Figures 3(a) and 3(b),

$$v(\tilde{q}(p, r); p, r) \leq 0 \text{ if and only if } \tilde{q}(p, r) \leq \gamma \text{; and}$$
$$v(q; p, r) \text{ is increasing in } q.$$

Therefore, if $\tilde{q}(p, r) < \gamma$, then as captured by the red dotted-line in Figure 3(a), the seller's profit is minimized when the probability of the seller receiving a signal $\tilde{q}(p, r)$ is maximized. The seller then can improve his guaranteed profit by offering a less generous refund, and thereby increasing the marginal signal.

In contrast, if the marginal signal is sufficiently high so that $\tilde{q}(p, r) > \gamma$, then the buyer who buys the product always brings a positive (ex-post) profit to the seller. The seller's profit is thus minimized when the probability of the seller receiving a signal above $\tilde{q}(p, r)$ is minimized, as captured by the red dotted line in Figure 3(b). The seller then can improve his guaranteed profit by offering a more generous refund, and thereby by lowering the marginal signal.

Combining these observations, we conclude that if the seller were to offer a positive refund, then he should set the price p as close to one as possible, and r to $\frac{p-\gamma}{1-\gamma}$ so that the marginal signal is exactly at γ. Hence the profit the seller can guarantee himself by offering a positive refund is $\frac{\pi-\gamma}{1-\gamma}$. Recall that the seller's best guaranteed profit without refund is $\left(1 - \sqrt{1-\pi}\right)^2$, as derived in (1). Thus, the seller's best guaranteed profit when the buyer may choose a signal distribution at $t = 3$ is

$$V_3^* = \sup_{p,r} \underline{V}_3(p, r) = \begin{cases} \frac{\pi-\gamma}{1-\gamma} & \gamma \leq \widehat{\gamma}(\pi) \\ \left(1 - \sqrt{1-\pi}\right)^2 & \gamma > \widehat{\gamma}(\pi) \end{cases}, \tag{3}$$

where $\widehat{\gamma}(\pi) \equiv \frac{2\left(1-\sqrt{1-\pi}\right)}{2-\sqrt{1-\pi}}$. Furthermore, the best guaranteed profit monotonically converges

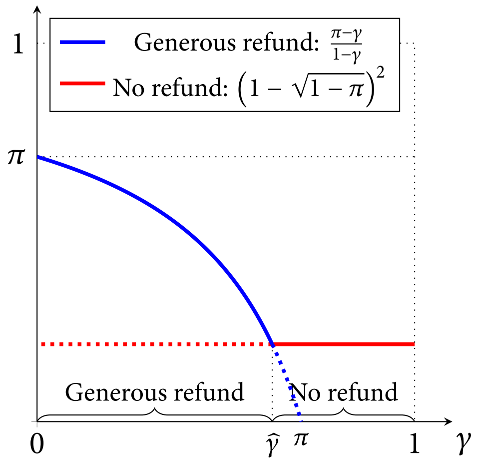


Figure 4: Best guaranteed profit V_3^*

to full surplus from trade π as the restocking cost converges to 0. The resulting bound is depicted on Figure 4.

Theorem 1. *Suppose that the buyer may choose a signal distribution at* $t = 3$*. Then the seller's best guaranteed profit is* V_3^* *defined in* (3)*. For any* $\varepsilon > 0$*, the seller's can achieve* $(1-\varepsilon) V_3^*$ *either by offering a generous refund* $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ *(when* $\gamma \leq \widehat{\gamma}(\pi)$*); or no refund* $(p, r) = \left(1-\sqrt{1-\pi}, 0\right)$ *(when* $\gamma > \widehat{\gamma}(\pi)$*).*

3.2 Policy-Independent Signal

We now consider the case where the buyer may choose a signal distribution only at $t = 1$. Recall that given an offer (p, r), the marginal signal is $\tilde{q}(p, r) = \frac{p-r}{1-r}$. Therefore,

$$V_1((p, r)|F) = p(1 - F(\tilde{q}(p, r))) - \mathbf{1}_{r>0}(c + r) \int_{\tilde{q}(p,r)}^{1} (1 - q) dF(q).$$

Our goal is to identify $V_1^* \equiv \sup_{(p,r)} \min_{F \in \mathcal{F}} V((p, r)|F)$. To this end, we introduce a few notations. First, to identify the profit from a non-refundable offer, define

$$G_V(q) \equiv \begin{cases} 0 & q \in [0, V) \\ 1 - \frac{V}{q} & q \in [V, 1) \\ 1 & q = 1 \end{cases}. \tag{4}$$

Then, for a given buyer's signal distribution F, the highest profit the seller can obtain by a non-refundable offer is identified by $\inf\{V : F(q) \geq G_V(q) \text{ for all } q\}$, which we denote by $V_{NR}(F)$.[11] Also define

$$F_V(q) \equiv \begin{cases} G_V(q) & q \in [0, \gamma) \\ 1 - \Phi(V) & q \in [\gamma, 1) \\ 1 & q = 1 \end{cases}. \tag{5}$$

[11] This comes from the same argument as in Roesler and Szentes (2017). The seller's profit from offering $(p, 0)$ is $V((p, 0)|F) = p(1 - F(p) + \Delta F(p))$. Furthermore, since $F(q) \geq G_{V_{NR}}(q)$ for all q, we have $V((p, 0)|F) \leq p(1 - G_{V_F}(p)) = V_{NR}(F)$. Therefore, $\sup_p V((p, 0)|F) = V_{NR}(F)$.

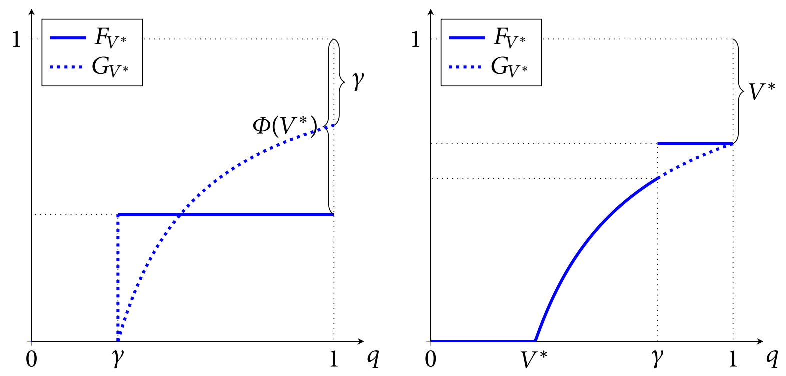
where

$$\Phi(V) = \begin{cases} \frac{V\left(\ln\frac{V}{\gamma}-1\right)+\pi}{1-\gamma} & V \in [0,\gamma) \\ \frac{\pi-\gamma}{1-\gamma} & V \in [\gamma,1) \end{cases}.$$

Notice that $\int_0^1 F_V(q)\,dq = 1-\pi$. Therefore, if the buyer's signal distribution is F_V, then $\Phi(V)$ is (the supremum of) the seller's profit from a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$. Observe that since $\Phi(V)$ is strictly decreasing in V on $[0,\gamma)$, and $\lim_{V\to\gamma}\Phi(V) = \frac{\pi-\gamma}{1-\gamma}$, either (i) $\Phi(V) > V$ for all $V \in [0,\gamma)$ (when $\frac{\pi-\gamma}{1-\gamma} > \gamma$, or equivalently $\gamma < 1-\sqrt{1-\pi}$), or (ii) there exists unique $\tilde{V} = \frac{\pi-\tilde{V}}{1-\gamma-\ln\frac{\tilde{V}}{\gamma}} \in [0,\gamma]$ such that $\Phi(\tilde{V}) = \tilde{V}$ (when $\gamma \geq 1-\sqrt{1-\pi}$). Let V^* be the largest $V \in [0,\gamma]$ such that $\Phi(V) \geq V$, i.e.,

$$V^* \equiv \begin{cases} \gamma & \gamma < 1-\sqrt{1-\pi} \\ \tilde{V} & \gamma \geq 1-\sqrt{1-\pi} \end{cases}.$$

Figuires 5(a) and 5(b) illustrate F_{V^*} when $\gamma < 1-\sqrt{1-\pi}$ and $\gamma \geq 1-\sqrt{1-\pi}$, respectively.



(a) $\gamma < 1-\sqrt{1-\pi}$ (b) $\gamma \geq 1-\sqrt{1-\pi}$

Figure 5: F_{V^*}

It is straightforward to observe that $\Phi(V^*) = \sup_{(p,r)} V((p,r)|F_{V^*})$ by construction of F_{V^*}. Below, we show that

$$\sup_{(p,r)} V((p,r)|F) \geq \sup_{(p,r)} V((p,r)|F_{V^*}) \text{ for all } F \in \mathcal{F}.$$

That is, $V_1^* = \Phi(V^*)$. To see this, first suppose there exists a distribution F such that $F(q) < F_{V^*}(q)$ for some $q \in [0,\gamma)$. Then, since $V_{NR}(F) > V^*$, we have $\sup_{(p,r)} V((p,r)|F) > \sup_{(p,r)} V((p,r)|F_{V^*}(q))$. Next, suppose that $F(q) \geq F_{V^*}(q)$ for all $q \in [0,\gamma)$. Recall that when the seller offers a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$, the buyer buys if and only if $q \geq \gamma$; and the seller's profit from the buyer with signal q is $(1-\varepsilon)\frac{q-\gamma}{1-\gamma}$. Therefore, the seller's profit from a generous refund when the buyer's signal distribution is F is (weakly) higher than $(1-\varepsilon)\,\Phi(V^*)$. Consequently, $\sup_{(p,r)} V((p,r)\,|F) \ge \sup_{(p,r)} V((p,r)\,|F_{V^*}(q))$.

We note that

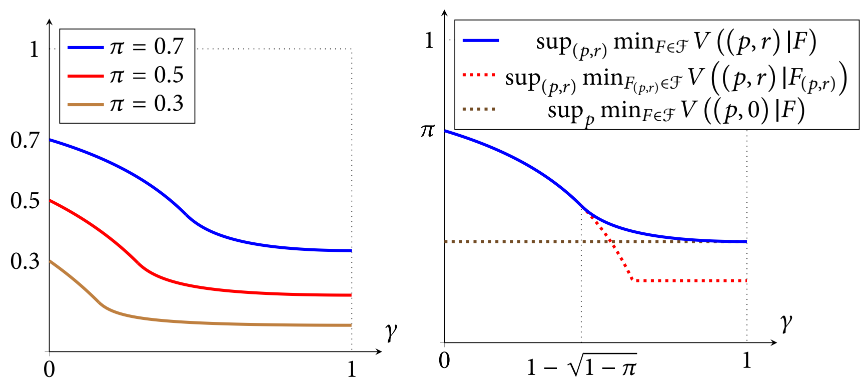
$$V_1^* \equiv \Phi(V^*) = \max_{V\in[0,\gamma]} \frac{\pi - V}{1-\gamma-\ln\frac{V}{\gamma}}, \tag{6}$$

and hence[12]

$$V_1^* = \begin{cases} \frac{\pi-\gamma}{1-\gamma} & \text{if } \gamma < 1-\sqrt{1-\pi} \\ \frac{\pi - V_1^*}{1-\gamma-\ln\frac{V_1^*}{\gamma}} & \text{if } \gamma \ge 1-\sqrt{1-\pi} \end{cases}. \tag{7}$$

We therefore have the following theorem.

Theorem 2. *Suppose that the buyer may choose signal distribution at $t=1$. Then, the best guaranteed profit for the seller V_1^* is defined by* (7).



(a) V_1^* for various priors

(b) V_1^* (blue) and V_3^* (red)

Figure 6: Best guaranteed profit V_1^*

A few observations follow. As one can see in Figure 6(a), the seller's best guaranteed profit strictly increases as the prior π goes up (for a given restocking cost γ); and as the restocking cost γ goes down (for a given prior π).

More importantly, the best guaranteed profit when the seller can only make non-refundable offers, $\sup_p \min_{F\in\mathcal{F}} V((p,0)\,|F)$ (i.e., the brown dotted-line in Figure 6(b)), is strictly lower than $V_1^* = \sup_{(p,r)} \min_{F\in\mathcal{F}} V((p,r)\,|F)$ for any level of restocking cost γ. That is, being able to offer a refund strictly improves the seller's best guaranteed profit. This is a clear contrast to the case where the buyer may choose a signal distribution in $t=3$, where the refund improves the best guaranteed profit if and only if the (normalized) restocking cost γ is sufficiently small, i.e., $\gamma < \hat{\gamma}(\pi)$.

Furthermore, when the restocking cost γ is small, i.e., $\gamma < 1-\sqrt{1-\pi}$, whether the buyer may choose a signal distribution at $t=1$ or $t=3$ does not affect the seller's best

[12] When $\gamma \ge 1-\sqrt{1-\pi}$, $V_1^* = \frac{-\pi}{W_{-1}\left(-\frac{\pi}{\gamma}e^{\gamma-2}\right)}$, where $W_{-1}(\cdot)$ denotes the lower branch of the Lambert's W.

guaranteed profit. That is, the blue-line and the red dotted-line in Figure 6(b) coincide. This is because when γ is small, or equivalently the prior π is sufficiently large, the cost of offering a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ is not significant, and hence the seller can achieve the best guaranteed profit in both cases by offering a generous refund. Recall that the worst signal distribution for a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ when the signal distribution is offer-dependent is F_{V^*}.[13] That is, not only is F_{V^*} depicted in Figure 5(a) the worst-signal distribution when the distribution is offer-dependent, but also when the distribution is policy-idenepdent.

We now construct the seller's strategy that achieves V_1^* when the restocking cost is not small, i.e., $\gamma \geq 1-\sqrt{1-\pi}$ as in Figure 5(b). We note that for any seller's pure strategy (p, r) such that $V\left((p, r) | F_V^*\right) \geq V_1^* - \varepsilon$, there exists a signal distribution F such that $V_1^* > V\left((p, r) | F\right)$. That is, the seller cannot guarantee V_1^* by making a deterministic offer. However, we show that the seller can guarantee himself a profit arbitrary close to V_1^* by the following policy $S^{\varepsilon}(p, r)$, which we call *the exponential pricing policy with a generous refund*: the seller makes (i) offers without refunds $(p, 0)$ on $p \in \left[V_1^*, \gamma\right)$ with density $s_0^{\varepsilon}(p) = \frac{1}{p\left(1-\gamma-\ln\frac{V_1^*}{\gamma}\right)}$; and (ii) a generous refund $(p, r) = \left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ with probability $s_r = 1-\int_{V_1^*}^{\gamma} s_0^{\varepsilon}(p)\, dp = \frac{1-\gamma}{1-\gamma-\ln\frac{V_1^*}{\gamma}}$.[14]

Theorem 3. *The seller can guarantee himself a profit that is arbitrary close to V_1^* by using the exponential pricing policy with with a generous refund $S^{\varepsilon}(p, r)$ for a sufficiently small ε.*

Proof. In the appendix. □

When the seller uses the exponential pricing with refund, and the buyer's signal distribution is F, by the proof of Theorem 3, the seller's profit from the non-refundable offers, and the profit from refundable offer, are respectively,

$$V_{NR}(F) \equiv \frac{\gamma - V_1^* - \int_{V_1^*}^{\gamma} F(p)\, dp}{1-\gamma+\ln\frac{V_1^*}{\gamma}} \geq \frac{\gamma - V_1^* - \int_0^{\gamma} F(p)\, dp}{1-\gamma+\ln\frac{V_1^*}{\gamma}}, \text{ and}$$

$$V_R(F) \equiv \frac{\pi - \gamma + \int_0^{\gamma} F(p)\, dp}{1-\gamma+\ln\frac{V_1^*}{\gamma}}.$$

Also recall that $V_1^* = \frac{\pi - V_1^*}{1-\gamma+\ln\frac{V_1^*}{\gamma}}$ by (7). Observe that since $\int_0^1 F(p)\, dp = 1-\pi$, $\int_0^{\gamma} F(p)\, dp$ is large when the buyer's signal distribution F is likely to generate a signal above γ, i.e., when F is informative. The seller's profit from the refundable offer is large when the signal distribution F is sufficiently informative, i.e., $V_R(F)$ is increasing in $\int_0^{\gamma} F(p)\, dp$. The seller's profit from the non-refundable offers may or may not be small. But even when it is small, the large profit from the refundable offer always compensates the forgone profits from the non-refundable offers.

Similarly, $\int_0^{\gamma} F(p)\, dp$ tends to be small when the buyer's signal distribution F is not so informative that F is unlikely to generate a signal sufficiently higher than γ. Then,

[13] See Figure 3.

[14] This is a variant of mechanism found in Du (2018).

the seller's profit from the refundable offer is small. However, the profit from the non-refundable offer is sufficiently large so that it compensates the small profit from the refundable offer.

This is the reason that the exponential pricing with refund can guarantee the seller at least V_1^* no matter the what the buyer's signal distribution is. This also explains why the seller's guaranteed profit V_1^* is strictly higher than $\sup_p \min_{F\in\mathcal{F}} V\left((p,0)\,|F\right)$ (i.e., the brown dotted-line in Figure 6(b)). The seller needs to offer a sufficiently low price if he wants to protect himself from the signal distributions that are sufficiently informative without using a refundable offer. Then, however, such a pricing policy results results in a low profit when the buyer's signal distribution is not so informative.[15]

3.3 Policy-Dependent Signal

We now consider the case where the buyer may choose a signal distribution only at $t = 2$. Our goal is to identify

$$V_2^* \equiv \sup_{\Delta\{(p,r)\}} \min_{F_{\Delta\{(p,r)\}}\in\mathcal{F}} \mathbb{E}_{\Delta\{(p,r)\}} V\left((p,r)\,|F_{\Delta\{(p,r)\}}\right).$$

As an immediate corollary of Theorem 3, we have the following theorem.

Theorem 4. *Suppose that the buyer may choose a signal distribution only at $t = 2$. Then, for any restocking cost γ, $V_1^* = V_2^*$. The seller can guarantee himself a profit that is arbitrary close to V_1^* either by (i) offering a generous refund (when $\gamma \le 1 - \sqrt{1-\pi}$); or (ii) using the exponential pricing policy with a generous refund $S^\varepsilon(p,r)$ for a sufficiently small ε (when $\gamma > 1 - \sqrt{1-\pi}$).*

3.4 Direct Mechanism

We have identified the best guaranteed profit that the seller can achieve using a combination of uniform prices and uniform prices with refunds. One may wonder if the seller can improve his best guaranteed profit by using a more intricate mechanism that potentially screens the buyer based on her signal q. Below we show that there exists no such a mechanism.

To this end, consider the environment where the buyer may choose a signal distribution only at $t = 1$. We first note that, for any mechanism, there exists an outcome-equivalent simple static direct mechanism with refunds.

Definition 1. *We say a direct mechanism $M \equiv \{p(q), \{\alpha_0(q), \alpha_r(q)\}\}_{q\in[0,1]}$ is a direct mechanism with refunds if, for each buyer's signal $q \in [0,1]$, the mechanism specifies (i) $p(q)$: the transfer from the buyer to the seller; (ii) $\alpha_0(q) \in [0,1]$: the probability that the*

[15]The result reported here is closely related to the finding in Du (2018) that analyzes the environment where the seller cannot make a refundable offer (or equivalently the restocking cost is $c = \infty$). Du (2018) shows that the distribution $F_{\tilde{V}}$, where $\tilde{V} \in (0, V_1^*)$ that uniquely solves $\tilde{V}(1 - \ln\tilde{V}) = \mu$, is the worst possible buyer's information structure for the seller; and that the exponentially randomized prices over $[\tilde{V}, 1]$ achieves the seller's best guaranteed profit regardless of the buyer's signal distribution. We also note that Roesler and Szentes (2017) shows that this distribution $F_{\tilde{V}}$ maximizes the buyer's welfare when the seller best responds to this information structure via uniform pricing.

buyer receives the product without an option to return; and (iii) $\alpha_r(q) \in [1-\alpha_0(q), 1]$*: the probability that the buyer receives the product with an option to return with refund* $r = 1$.

Lemma 1. *For any outcome the seller can induce by an indirect mechanism, there exists an outcome-equivalent direct mechanism with refunds that is individually rational and incentive compatible.*

Proof. In the appendix. □

Under the direct mechanism with refunds M, if the buyer with signal q reports q', her payoff is

$$\begin{aligned} U\left(q';q|M\right) &\equiv \left(\alpha_0\left(q'\right)+\alpha_r\left(q'\right)\right)q+\alpha_r\left(q'\right)(1-q)-p(q) \\ &= q\times\alpha_0\left(q'\right)-\left(p(q)-\alpha_r\left(q'\right)\right). \end{aligned}$$

The seller's profit from the buyer with signal q is

$$\begin{aligned} R(q|M) &\equiv p(q)-\alpha_r\left(q'\right)(1-q)(c+1) \\ &= p(q)-\alpha_r\left(q'\right)\frac{1-q}{1-\gamma}, \end{aligned}$$

and hence his profit when the buyer's signal distribution is F is

$$V(M;F) \equiv \int_0^1 R(q|M)\,dF(q).$$

Thus seller's objective is to find a mechanism $M = \{p(q), \{\alpha_0(q), \alpha_r(q)\}\}_{q\in[0,1]}$ that solves

$$\max_{M\in\mathcal{M}}\min_{F\in\mathcal{F}} V(M;F) \tag{8}$$

where $\mathcal{M}$ is the set of all direct mechanisms with refunds M that satisfy the following two conditions:

$$\begin{aligned} &\text{IC: } U(q;q|M) \geq U\left(q';q|M\right) \text{ for all } q' \text{ and } q \\ &\text{IR: } U(q;q|M) \geq 0 \text{ for all } q. \end{aligned}$$

By adopting the standard argument, we can simplify the seller's problem to the one in which he only chooses an increasing function $\alpha_0(\cdot)$ (instead of M, which is a triplet of functions, that is individually rational and incentive compatible). To formally state this result, for a function $\alpha_0 : [0,1] \to [0,1]$, define $\underline{R}(\pi;\alpha_0) \equiv -\text{con}\left[-R(q;\alpha_0)\right](\pi)$, where

$$R(q;\alpha_0) \equiv \begin{cases} q\times\alpha_0(q)-\int_0^q \alpha_0(\tilde{q})\,d\tilde{q} & \text{if } q<\gamma \\ q\times\alpha_0(q)+\frac{q-\gamma}{1-\gamma}(1-\alpha_0(q))-\int_0^q \alpha_0(\tilde{q})\,d\tilde{q} & \text{if } q\geq\gamma \end{cases}. \tag{9}$$

We then have:

Lemma 2. *Take* $\alpha_0^*(q) \in \arg\max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0)$*, where* $\mathcal{A}$ *is the set of all increasing functions from* $[0,1]$ *to* $[0,1]$*. Then* $M^* = \{p^*(q), \{\alpha_0^*(q), \alpha_r^*(q)\}\}_{q \in [0,1]}$*, where* $\alpha_r^*(q) = \mathbf{1}_{q \in [\gamma,1]} \times (1 - \alpha_0(q))$*, and* $p^*(q) = q\alpha_0^*(q) + \alpha_r^*(q) - \int_0^q \alpha_0^*(\tilde{q})\, d\tilde{q}$*, is a solution to the problem* (8)*. Furthermore, there exists* $V \in [0, \gamma]$ *such that*

$$R(q; \alpha_0^*) = \begin{cases} 0 & \text{if } q < V \\ \frac{q - V}{1 - \gamma - \ln \frac{V}{\gamma}} & \text{if } q \geq V \end{cases}.$$

and the seller's best guaranteed profit is

$$\max_{M \in \mathcal{M}} \min_{F \in \mathcal{F}} V(M; F) = \max_{V \in [0,\gamma]} \frac{\pi - V}{1 - \gamma - \ln \frac{V}{\gamma}}.$$

Proof. In Appendix. □

Theorem 5. $\max_{M \in \mathcal{M}} \min_{F \in \mathcal{F}} V(M; F) = V_1^* = V_2^*$*. That is, the best guaranteed profit under any mechanism is bounded from above by* V_1^**.*

Proof. It is straightforward to verify that $\frac{\pi - V}{1 - \gamma - \ln \frac{V}{\gamma}}$ is maximized at $V = V_1^*$. Furthermore $V_1^* = \frac{\pi - V_1^*}{1 - \gamma - \ln \frac{V_1^*}{\gamma}}$ by (6). Therefore, we have the required result. □

3.5 Feasible Outcomes

In this section, we identify the information structure that maximizes the buyer's welfare, as well as outcomes that can be supported by some information structure.

We first characterize the buyer-optimal outcome. Without loss of generality, we assume that the buyer chooses an information structure at $t = 1$ in Figure 1, and observe a signal at $t = 3$. The buyer's choice becomes public information, and it is assumed that the buyer cannot change the information structure she has chosen after $t = 1$.

Given our focus on the buyer-optimal outcome, we assume that when the seller is indifferent between two or more offers, the seller makes an offer that induces a higher buyer's payoff.

Then there are two possible scenarios that results in different buyer's payoffs.

Policy-independent signal The buyer chooses a single signal distribution $F \in \mathcal{F}$ at $t = 1$. The seller chooses an offer (p, r) to maximize his profit.

Offer-dependent signal The buyer chooses a set of signal distributions $\{F_{(p,r)}\}_{(p,r)}$ at $t = 1$ that maps each possible seller's offer (p, r) into a possibly different signal distribution $F_{(p,r)} \in \mathcal{F}$. The seller chooses an offer (p, r) to maximize his profit.

More formally, let $U((p, r) | F)$ be the buyer's payoff when her signal distribution at $t = 3$ is $F \in \mathcal{F}$, and the seller's offer is (p, r). Also, let $V((p, r) | F)$ be the seller's profit from offer (p, r) and the buyer's signal distribution after $t = 3$ is F. Then, the buyer-optimal outcomes when the signals are, policy-independent, and offer-dependent, are respectively characterized as

$$(10)\qquad \begin{aligned} U_1^* &\equiv \max_{F\in\mathcal{F}} U\left((p_1,r_1)\,|F\right) \text{ s.t. } (p_1,r_1)\in\arg\max_{(p,r)} V\left((p,r)\,|F\right)\\ U_3^* &\equiv \max_{F_{(p,r)}\in\mathcal{F}} U\left((p_3,r_3)\,|F_{(p_3,r_3)}\right) \text{ s.t. } (p_3,r_3)\in\arg\max_{(p,r)} V\left((p,r)\,|F_{(p,r)}\right)\end{aligned}$$

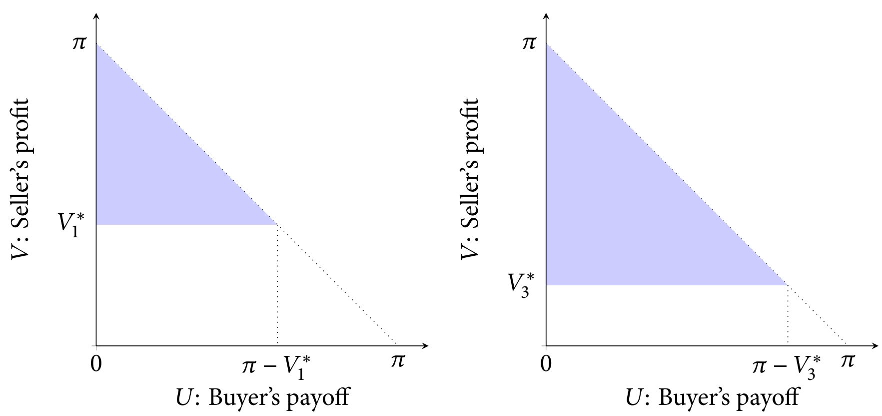
Our goal is to identify U_1^* and U_3^*. Before proceeding to with the analysis, we make two observations. First, $U_1^* \le U_3^*$. Furthermore, the total surplus is bounded from above by π. However, the preceding analysis identifies that the seller's best guaranteed profit is V_1^* when the signal is policy-independent (Theorem 2); and V_3^* $(\le V_1^*)$ when the signal is offer-dependent (Theorem 1). Therefore,

$$U_1^* \le \pi - V_1^* \text{ and } U_3^* \le \pi - V_3^*.$$

The theorem below shows that the buyer can obtain the respective upper bounds.

Theorem 6. *If the buyer can choose and commit to a policy-independent signal distribution at $t = 1$, then her payoff under the buyer-optimal outcome is $U_1^* = \pi - V_1^*$. If she can choose and commit to a set of offer-depend signal distributions at $t = 1$, then, her payoff under the buyer-optimal outcome is $U_3^* = \pi - V_3^*$.*

Proof. In the Appendix. □



(a) O_1: Policy-dependent signal (b) O_3: Offer-dependent signal

Figure 7: Feasible outcomes

We note that there exist multiple policy-independent signal distributions (or sets of offer-dependent signal distributions) that support result in a same buyer-optimal outcomes.[16]

Also, as an immediate corollary, we can characterize pairs of buyer's payoff and seller's profit that can be supported by some information structure. More precisely, we say $\left(\hat{U},\hat{F}\right)$

[16]An alternative interpretation of our buyer-optimality result is as follows: the buyer ex-ante delegate the information gathering to a third-party knowing that the seller best responds to the information structure by choosing a price-refund pair. Our best-guaranteed profit is a constraint for buyer optimality.

is a *feasible outcome* supported by a policy-independent signal F if $(\hat{p}, \hat{r})$ maximizes $V((p,r)|F)$ with respect to (p,r); $\hat{V} = V((\hat{p},\hat{r})|F)$; and $U = \hat{U}((\hat{p},\hat{r})|F)$. Similarly, we can define a feasible outcome supported by a set of offer-dependent signal distributions. Then, the set of feasible outcomes O_1 supported by some policy-independent signals; and the set of outcomes O_3 supported by some set of offer-dependent signal distributions are, respectively,

$$O_1 \equiv \{(U,V) : V \in [V_1^*, \pi - U], U \in [0, \pi - V_1^*]\}$$
$$\text{and } O_3 \equiv \{(U,V) : V \in [V_3^*, \pi - U], U \in [0, \pi - V_3^*]\}.$$

4 Discussion

We analyzed the seller's pricing decision with refunds when the seller does not know about the information the buyer has. One aspect that we did not address is the learning about demand through pricing. Our insights that well-designed refund policy limits the significance of buyer learning on the seller's profit should carry out even in a dynamic environment. However, in a dynamic environment, carefully designed dynamic prices and resulting buyer's purchasing and return decisions can be used to improve future pricing decisions. Investigating how such seller's learning motive would shape the intertemporal pricing with refunds would be an interesting venue for future research.

Another venue for the research is the application to platform designs. A platform, e.g., eBay or Airbnb, could choose information that is revealed to the buyer. The seller chooses pricing and return policy, and the buyer makes an purchasing and return decision. In the absence of competition among platforms, it is natural to expect that one of the Pareto-optimal outcomes identified in the previous section would arise. Analyzing how the competition affects the welfare and equilibrium information structure is an interesting question.

References

Au, P. H. and K. Kawai (2017a): "Competitive Disclosure of Correlated Information," *Working Paper, UNSW*.

——— (2017b): "Competitive Information Disclosure by Multiple Senders," *Working Paper, UNSW*.

Aumann, R., M. Maschler, and R. Stearns (1995): *Repeated Games with Incomplete Information*, MIT Press.

Boleslavsky, R. and C. Cotton (2018): "Limited Capacity in Project Selection: Competition through Evidence Production," *Economic Theory*, 65, 385–421.

Che, Y.-K. (1996): "Customer Return Policies for Experience Goods," *The Journal of Industrial Economics*, 44, 17–24.

Courty, P. and H. Li (2000): "Sequential Screening," *The Review of Economic Studies*, 67, 697–717.

Du, S. (2018): "Robust Mechanisms Under Common Valuation," *Working Paper, Simon Fraser University*.

Ely, J. C., D. Garrett, and T. Hinnosaar (2017): "Overbooking," *Journal of the European Economic Association*, 15, 1258–1301.

Escobari, D. and P. Jindapon (2014): "Price Discrimination through Refund Contracts in Airlines," *International Journal of Industrial Organization*, 34, 1–8.

Eso, P. and B. Szentes (2007): "Optimal Information Disclosure in Auctions and the Handicap Auction," *Review of Economic Studies*, 74, 705–731.

Gale, I. L. and T. J. Holmes (1992): "The Efficiency of Advance-Purchase Discounts in the Presence of Aggregate Demand Uncertainty," *International Journal of Industrial Organization*, 10, 413–437.

——— (1993): "Advance-Purchase Discounts and Monopoly Allocation of Capacity," *The American Economic Review*, 83, 135–146.

Gallego, G. and O. Sahin (2010): "Revenue Management with Partially Refundable Fares," *Operations Research*, 58, 817–833.

Grossman, S. J. (1981): "The Informational Role of Warranties and Private Disclosure about Product Quality," *The Journal of Law & Economics*, 24, 461–483.

Inderst, R. and M. Ottaviani (2013): "Sales Talk, Cancellation Terms and the Role of Consumer Protection," *The Review of Economic Studies*, 80, 1002–1026.

Inderst, R. and G. Tirosh (2015): "Refunds and Returns in a Vertically Differentiated Industry," *International Journal of Industrial Organization*, 38, 44–51.

Kamenica, E. and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

Krahmer, D. and R. Strausz (2015): "Optimal Sales Contracts with Withdrawal Rights," *The Review of Economic Studies*, 82, 762–790.

Libgober, J. and X. Mu (2017): "Informational Robustness in Intertemporal Pricing ," *Working Paper, Harvard University*.

Moorthy, S. and K. Srinivasan (1995): "Signaling Quality with a Money-Back Guarantee: The Role of Transaction Costs," *Marketing Science*, 14, 442–466.

Nocke, V., M. Peitz, and F. Rosar (2011): "Advance-Purchase Discounts as a Price Discrimination Device," *Journal of Economic Theory*, 146, 141–162.

Roesler, A.-K. and B. Szentes (2017): "Buyer-Optimal Learning and Monopoly Pricing," *American Economic Review*, 107, 2072–80.

Schmalense, R. (1981): "Monopolistic Two-Part Pricing Arrangements," *The Bell Journal of Economics*, 12, 445–466.

Zhang, J. (2013): "Revenue Maximizing with Return Policy when Buyers Have Uncertain Valuations," *International Journal of Industrial Organization*, 31, 452–461.

A Proofs

Proof of Theorem 3: When F is the buyer's information structure, the seller's payoff of using policy S^{ε} is

$$\mathbb{E}_{S^{\varepsilon}}\left[V\left((p,r)\,|F\right)\right] = \int_{V_1^*}^{\gamma} p\left(1-F\left(p\right)\right)s_0^{\varepsilon}\left(p\right)dp + s_r\int_{\gamma}^{1}\left((1-\varepsilon)-(1-q)\left(\left(1-\frac{\varepsilon}{1-\gamma}\right)+c\right)\right)dF\left(q\right).$$

Then,

$$\int_{V_1^*}^{\gamma} p\left(1-F\left(p\right)\right)s_0^{\varepsilon}\left(p\right)dp = \frac{1}{1-\gamma+\ln\frac{V_1^*}{\gamma}}\int_{V_1^*}^{\gamma}\left(1-F\left(p\right)\right)dp \geq \frac{\gamma-V_1^*-\int_0^{\gamma}F\left(p\right)dp}{1-\gamma+\ln\frac{V_1^*}{\gamma}},$$

and

$$\begin{aligned}
&s_r\int_{\gamma}^{1}\left((1-\varepsilon)-(1-q)\left(\left(1-\frac{\varepsilon}{1-\gamma}\right)+c\right)\right)dF\left(q\right)\\
&=\frac{1-\gamma}{1-\gamma+\ln\frac{V_1^*}{\gamma}}\left(1-(1+c)\int_{\gamma}^{1}F\left(q\right)dq-\varepsilon\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)\right)\\
&=\frac{1-\gamma-\int_{\gamma}^{1}F\left(q\right)dq}{1-\gamma+\ln\frac{V_1^*}{\gamma}}-\frac{\varepsilon\left(1-\gamma\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)\\
&=\frac{\pi-\gamma+\int_0^{\gamma}F\left(p\right)dp}{1-\gamma+\ln\frac{V_1^*}{\gamma}}-\frac{\varepsilon\left(1-\gamma\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)
\end{aligned}$$

We thus have

$$\begin{aligned}
\mathbb{E}_{S^{\varepsilon}}\left[V\left((p,r)\,|F\right)\right] &\geq \frac{\pi-V_1^*}{1-\gamma-\ln\frac{V_1^*}{\gamma}}-\frac{\varepsilon\left(1-\gamma\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)\\
&= V_1^*-\frac{\varepsilon\left(1-\gamma\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right).
\end{aligned}$$

We thus have the required result.

Proof of Lemma 1: Any outcome the seller can induce by an indirect mechanism can be induced by an individually-rational and incentive-compatible direct mechanism $\Phi = \left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$ that specifies, for each $q \in [0,1]$, (i) α_q: the probability that the buyer receives the product; (ii) p_q: the transfer from the buyer to the seller; and (iii) $\left\{\kappa_q^v, \tau_q^v\right\}_{v\in\{0,1\}}$: the direct mechanism that specifies, for each buyer's true valuation $v \in \{0,1\}$, (a) κ_q^v: the probability the buyer keeps the product; and (b) τ_q^v: the transfer from the seller to the buyer with the following properties: (IC) $\kappa_q^1 + \tau_q^1 \geq 1$ and $\kappa_q^0 + \tau_q^0 \geq 0$ and (IR) $\kappa_q^1 + \tau_q^1 \geq \kappa_q^0 + \tau_q^0$ and $\tau_q^0 \geq \tau_q^1$. Notice that for any stochastic direct mechanisms (over $\left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$), there exists an outcome-equivalent deterministic direct mechanism. So without loss, we limit our attention to the deterministic mechanisms.

Under this direct mechanism $\Phi = \left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$, the payoff of the buyer with signal q and the seller's profit from her are, respectively,

$$\begin{aligned} u\left(\Phi\right) &\equiv \alpha_q\left(q\left(\kappa_q^1 + \tau_q^1\right) + (1-q)\,\tau_q^0\right) - p_q, \\ v\left(q|\Phi\right) &\equiv p_q - \alpha_q\left[q\left(\left(1-\kappa_q^1\right)c + \tau_q^1\right) - (1-q)\left(\left(1-\kappa_q^0\right)c + \tau_q^0\right)\right]. \end{aligned}$$

Notice that without loss, we can assume that $\kappa_q^1 = 1$, $\tau_q^1 = 0$, $\tau_q^0 = 1$. To see this, consider $\tilde{\Phi} \equiv \left\{\alpha_q, \tilde{p}_q, \left\{\left(\tilde{\kappa}_q^0, \tilde{\tau}_q^0\right), \left(\tilde{\kappa}_q^1, \tilde{\tau}_q^1\right)\right\}\right\}$ such that $\left(\tilde{\kappa}_q^1, \tilde{\tau}_q^1\right) \neq (1,0)$; and $\Phi \equiv \left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$ such that (i) $\left(\kappa_q^1, \tau_q^1\right) = (1,0)$; (ii) $\left(\kappa_q^0, \tau_q^0\right) = \left(\tilde{\kappa}_q^0, 1\right)$; (iii) $p_q = \tilde{p}_q - \alpha_q q\left(\tilde{\kappa}_q^1 + \tilde{\tau}_q^1 - 1\right) + \alpha_q(1-q)\left(\tilde{\tau}_q^0 - 1\right)$. Then,

$$\begin{aligned} u\left(\tilde{\Phi}\right) &= \alpha_q\left(q\left(\tilde{\kappa}_q^1 + \tilde{\tau}_q^1\right) + (1-q)\,\tilde{\tau}_q^0\right) - \tilde{p}_q \\ &= \alpha_q\left(q + (1-q)\right) - p_q = u\left(\Phi\right) \end{aligned}$$

and

$$\begin{aligned} v\left(q|\tilde{\Phi}\right) &= \tilde{p}_q - \alpha_q\left[q\left(\left(1-\tilde{\kappa}_q^1\right)c + \tilde{\tau}_q^1\right) - (1-q)\left(\left(1-\tilde{\kappa}_q^0\right)c + \tilde{\tau}_q^0\right)\right] \\ &= p_q - \alpha_q\left[q\left(\left(1-\tilde{\kappa}_q^1\right)(c+1)\right) + (1-q)\left(\left(1-\tilde{\kappa}_q^0\right)c + 1\right)\right] \\ &\leq p_q - \alpha_q(1-q)\left(\left(1-\tilde{\kappa}_q^0\right)c + 1\right) = v\left(q|\Phi\right). \end{aligned}$$

If we denote $\alpha_0(q) = \alpha_q(1-q)\,\tilde{\kappa}_q^0$, and $\alpha_r(q) = \alpha_q\left(q + (1-q)\left(1-\tilde{\kappa}_q^0\right)\right)$ so that $\alpha_0(q) + \alpha_r(q) = \alpha_q$, then we have the required result.

Proof of Lemma 2: We first show that for any direct mechanism with refunds $\tilde{M} \in \mathcal{M}$, there exists another direct mechanism with refunds $M \in \mathcal{M}$ with the following properties: (i) $R\left(q|\tilde{M}\right) \leq R\left(q|M\right)$ for all q; (ii) $\alpha_0(q)$ is increasing, and (iii) $R\left(q|M\right) = R\left(q;\alpha_0\right)$.

By the standard argument, the incentive compatibility condition is equivalent to that

$$\alpha_0(q) \text{ is increasing in } q \text{ and } U\left(q;q|M\right) = \int_0^q \alpha_0\left(\tilde{q}\right)d\tilde{q}.$$

Since $\int_0^q \alpha_0\left(\tilde{q}\right)d\tilde{q} = q\alpha_0(q) - \left(p(q) - \alpha_r(q)\right)$,

$$\begin{aligned} R\left(q|M\right) &= p(q) - a_r(q)\frac{1-q}{1-\gamma} \\ &= q\alpha_0(q) + \alpha_r(q) - \int_0^q \alpha_0\left(\tilde{q}\right)d\tilde{q} - a_r(q)\frac{1-q}{1-\gamma} \\ &= q\alpha_0(q) + \frac{q-\gamma}{1-\gamma}\alpha_r(q) - \int_0^q \alpha_0\left(\tilde{q}\right)d\tilde{q}. \end{aligned}$$

Observe that $R(0) \leq 0$. Therefore, if $q = 0$, then the seller chooses $\alpha_0(q) = \alpha_r(q) = 0$. Next, if $q \in (0,\gamma)$, then since $\frac{q-\gamma}{1-\gamma} < 0$, $\alpha_r(q) = 0$. If $q = \gamma$, $R\left(\gamma|M\right)$ does not depend on $\alpha_r(q)$. Therefore, $\alpha_r(q) = 1 - \alpha_0(q)$. Similarly, if $q \in (\gamma, 1]$, then since $\frac{q-\gamma}{1-\gamma} > 0$, $\alpha_r(q) = 1 - \alpha_0(q)$. We thus can conclude that

$$\max_{M \in \mathcal{M}} \min_{F \in \mathcal{F}} V(M; F) = \max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0).$$

Take $\alpha_0^*(q) \in \arg\max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0)$, M^*, and the the lower linear envelope function of $\underline{R}(q; \alpha_0^*)$ at $q = \pi$ that has the steepest slope denoted by $L(q) \equiv \chi(q - V)$. Observe that $\text{supp}\{F^*\} \subset Q \equiv \{q : R(q; \alpha_0^*) = L(q)\}$ for any $F^* \in \arg\max_{F \in \mathcal{F}} \min_{F \in \mathcal{F}} V(M^*; F)$, and hence $\alpha_0(q)$ is strictly increasing at q only if $q \in \text{supp}\{F^*\}$. Therefore implies $R(q; \alpha_0^*) = \max\{0, L(q)\}$ for all $q \in [0, 1]$.

By (9), $R(q; \alpha_0^*) = \chi(q - V)$ on $q \in [V, 1]$ implies that for some k,

$$\alpha_0^*(q) = \begin{cases} \chi \ln \frac{q}{V} & q \in [V, \gamma] \\ 1 - (1-\gamma)\chi + k \times (-(1-q))^{-\frac{1}{\gamma}} & q \in [\gamma, 1] \end{cases}.$$

However, since $(-(1-q))^{-\frac{1}{\gamma}}$ is not real for any $q \in [\gamma, 1]$, we have $k = 0$. Thus $\chi \ln \frac{\gamma}{V} = 1 - (1-\gamma)\chi$, or $\chi = \frac{1}{1-\gamma-\ln\frac{V}{\gamma}}$, equivalently. We thus have $\max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0) = \max_{V \in [0,\gamma]} \frac{\pi - V}{1-\gamma-\ln\frac{V}{\gamma}}$.

Proof of Theorem 6: We start with the case where $\gamma \geq 1 - \sqrt{1-\pi}$. Consider $F_{V_1^*}(q)$, where F_V is defined in (5) and V_1^* in Theorem 2. $F_{V_1^*}(q)$ is illustrated in Figure 5(b), and $\sup_{(p,r)} V\left((p,r) | F_{V_1^*}\right) = V_1^*$. Notice that $V\left((V_1^*, 0) | F_{V_1^*}\right) = V_1^*$. Furthermore, since $\text{supp}\left\{F_{V_1^*}(q)\right\} = [V_1^*, \gamma] \cup \{1\}$, the buyer's payoff when $(p, r) = (V_1^*, 0)$ is $\pi - V_1^*$. We thus have $U_1^* = \pi - V_1^*$.

Next, we consider the case where $\gamma < 1 - \sqrt{1-\pi}$. Define

$$F^*(q) \equiv \begin{cases} 0 & q \in [0, V_1^*) \\ 1-\gamma & q = [V_1^*, 1) \\ 1 & q = 1 \end{cases}.$$

The seller's profit from offering $(V_1^*, 0)$ is $V_1^* = \frac{\pi-\gamma}{1-\gamma}$. The seller's profit from $(p, r), r > 0$ is bounded from V_1^*. To see this, notice that if $\tilde{q}(p, r) > V_1^*$, then

$$V((p,r) | F^*) = \gamma p \leq \gamma \leq \frac{\pi - \gamma}{1-\gamma} = V_1^*,$$

and if $\tilde{q}(p, r) \leq V_1^*$, then since $\tilde{q}(p, r) = \frac{p-r}{1-r}$,

$$\begin{aligned} V((p,r) | F^*) &= p - (1-\gamma)(1-\tilde{q}(p,r))\left(\frac{p - \tilde{q}(p,r)}{1-\tilde{q}(p,r)} + \frac{\gamma}{1-\gamma}\right) \text{ (increasing in } \tilde{q}(p,r)\text{)} \\ &\leq p - (1-\gamma)(1-V_1^*)\left(\frac{p - V_1^*}{1-V_1^*} + \frac{\gamma}{1-\gamma}\right) \text{ (increasing in } p\text{)} \\ &\leq 1 - (1-\gamma)(1-V_1^*)\left(\frac{1}{1-\gamma}\right) = V_1^*. \end{aligned}$$

Therefore, $(V_1^*, 0) \in \arg\max_{(p,r)} V((p,r) | F^*)$. Since $\text{supp}\{F^*(q)\} = \{V_1^*, 1\}$, the buyer's payoff when $(V_1^*, 0)$ is $\pi - V_1^*$.

Consider the following set of offer-dependent signal distributions $\left\{F^*_{(p,r)}\right\}$. If $(p, r) = (V_3^*, 0)$, then $F^*_{(p,r)}$ has an atom of size one at $q = \pi$, i.e., an uninformative signal; and if $(p, r) \neq (V_3^*, 0)$, then $F^*_{(p,r)} \in \arg\min_{F_{(p,r)} \in \mathcal{F}} V\left((p, r) | F_{(p,r)}\right)$. Then $V\left((V_3^*, 0) | F^*_{(V_3^*,0)}\right) = V_3^*$ and $V\left((p, r) | F^*_{(p,r)}\right) \leq V_3^*$ for all $(p, r) \neq (V_3^*, 0)$ by Theorem 1. Furthermore, under $F_{(V_3^*,0)}$, the buyer's posterior is $q = \pi > V_3^*$ the buyer's payoff is $\pi - V_3^*$.