Online Appendix to "On the impossibility of protecting risk-takers"*

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1 Proof of Lemma 1

Lemma 1. *Utility v is not asymptotically risk-loving if and only if there exists $\varepsilon > 0$ such that $v(t) \leq v(0) + \varepsilon t$ for all $t < 0$.*

Proof The result follows from three observations:

1. v is not asymptotically risk-loving if and only inf $\lim_{t\to-\infty} \frac{1}{t} v(t) \neq 0$, which is equivalent to $\lim_{t\to-\infty} \frac{v(0)-v(t)}{-t} > 0$.
2. By the monotonicity of the utility function, $v(0) > v(t)$ for all $t < 0$, so $\frac{v(0)-v(t)}{-t} > 0$ for all $t < 0$.
3. Finally, $\lim_{t\to 0} \frac{v(0)-v(t)}{-t} = v'(0) > 0$ by assumption.

Combining these three observations proves that v is not asymptotically risk-loving if and only if there exists $\varepsilon > 0$ such that

$$\frac{u(\theta, 0) - u(\theta, t)}{-t} \geq \varepsilon \iff u(\theta, t) \leq u(\theta, 0) + \varepsilon t \quad \forall t < 0. \tag{1}$$

□

2 Proof of Lemma 2

Lemma 2. *Utility v is asymptotically risk-loving*

1. *if* $\lim_{t\to-\infty} v'(t) = 0$ *or*
2. *if the utility function is bounded from below or*

*The paper itself is available at http://toomas.hinnosaar.net/impossibility.pdf

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3. *if the second derivative with respect to transfer $v''(t)$ exists and either*

$$\lim_{t\to-\infty}\frac{tv''(\theta,t)}{v'(\theta,t)} > 0 \text{ or } \lim_{t\to-\infty}\frac{v''(\theta,t)}{v'(\theta,t)} < 0 \tag{2}$$

(i.e., asymptotically risk-loving according to the Arrow-Pratt relative or absolute risk measure).

Proof of Lemma 2

1. By assumption, for any $\varepsilon > 0$, there exists $\hat{t} < 0$ such that $v'(t) < \varepsilon$ for all $t < \hat{t}$. Therefore, for all $t < \hat{t}$, $0 < v(\hat{t}) - v(t) < \varepsilon(\hat{t} - t)$; thus,

$$\frac{v(\hat{t})}{t} > \frac{v(t)}{t} > \frac{v(\hat{t}) - \varepsilon\hat{t}}{t} + \varepsilon \tag{3}$$

so that $\lim_{t\to-\infty}\frac{1}{t}v(t) \in [0,\varepsilon]$. By taking $\varepsilon \to 0$, $\lim_{t\to-\infty}\frac{1}{t}v(t) = 0$.

2. If there exists $\underline{v} < 0$ such that $v(t) \geq \underline{v}$ for all t, then

$$0 \leq \lim_{t\to-\infty}\frac{1}{t}v(t) \leq \lim_{t\to-\infty}\frac{1}{t}\underline{v} = 0. \tag{4}$$

3. Suppose first that $\lim_{t\to-\infty} -\frac{tv''(t)}{v'(t)} = \underline{r}^* > 0$ and fix $0 < \alpha < \underline{r}^*$. By Lemma 3, there exist $\hat{t} < 0$, $c_1 > 0$, and $c_2 \in \mathbb{R}$ such that for all $t < \hat{t}$, $v(t) > -c_1(-t)^{1-\alpha} + c_2$. Thus, $0 < \frac{1}{t}v(t) < -c_1(-t)^{-\alpha} + \frac{c_2}{t}$, which implies that $\lim_{t\to-\infty}\frac{1}{t}v(t) = 0$.

   Finally, $\lim_{t\to-\infty} -\frac{v''(t)}{v'(t)} = \underline{r} < 0$ implies $\lim_{t\to-\infty} -\frac{tv''(t)}{v'(t)} = \lim_{t\to-\infty} t\underline{r} = \infty > 0$.

□

Lemma 3. *Suppose* $\underline{r}^* = \lim_{t\to-\infty} -\frac{tv''(t)}{v'(t)}$.

1. *For any $\alpha < \underline{r}^* \leq 1$, there exist $\hat{t} < 0$, $c_1 > 0$, and $c_2 \in \mathbb{R}$ such that*

$$v(t) > -c_1(-t)^{1-\alpha} + c_2, \quad \forall t < \hat{t}. \tag{5}$$

2. *For any $1 > \alpha > \underline{r}^*$, there exist $\hat{t} < 0$, $c_1 > 0$, and $c_2 \in \mathbb{R}$ such that*

$$v(t) < -c_1(-t)^{1-\alpha} + c_2, \quad \forall t < \hat{t}. \tag{6}$$

Proof of Lemma 3 Suppose $\underline{r}^* > \alpha$. Following from the definition of $\underline{r}^*$, there exist $\hat{t}$ such that $-\frac{tv''(t)}{v'(t)} > \alpha$ or equivalently $\frac{v''(t)}{v'(t)} > -\frac{\alpha}{t}$ for all $t < \hat{t}$. Therefore,

$$\ln\frac{v'(\hat{t})}{v'(t)} = \int_t^{\hat{t}}\frac{v''(x)}{v'(x)}dx > -\alpha\ln\frac{\hat{t}}{t} \quad\Rightarrow\quad v'(t) < c_1(1-\alpha)(-t)^{-\alpha}, \tag{7}$$

where $c_1 = \frac{v'(\hat{t})}{(1-\alpha)(-\hat{t})^{-\alpha}} > 0$. Now for all $t < \hat{t}$,

$$v(\hat{t}) - v(t) < -c_1\left[(-\hat{t})^{1-\alpha} - (-t)^{1-\alpha}\right] \quad\Rightarrow\quad v(t) > -c_1(-t)^{1-\alpha} + c_2, \tag{8}$$

where $c_2 = v(\hat{t}) - c_1(-\hat{t})^{1-\alpha}$. Proof for $\alpha > \underline{r}^*$ is analogous.

□

3 Cumulative prospect theory preferences

Theorem 3. *Suppose that the bidders are asymptotically risk-loving cumulative prospect theory agents. Then there exists a non-random winner-pays auction where almost all types of buyers pay unboundedly large expected transfers.*

Proof Again, I divide the proof into two lemmas: First, Lemma 4 shows that for any T-mechanism, I can construct the function γ that makes truth-telling optimal for all types. Second, Lemma 5 shows that if buyers are asymptotically risk-loving, then with sufficiently large T, the T-mechanism ensures arbitrarily large expected transfers from each type and, therefore, unbounded profits.

□

Lemma 4. *For a big T, there exists function γ such that bidding one's own value is an equilibrium in T-mechanism.*

Proof Fix an arbitrary T-mechanism. The expected utility for a bidder with value θ_i who bids θ_i' is[1]

$$U(\theta_i'|\theta_i) = w^-(G(\theta_i') - G(\gamma(\theta_i')))u(\theta_i, -T) + w^+(G(\gamma(\theta_i'))u(\theta_i, 0), \tag{9}$$

The condition for the optimality of bidding one's own type is $\frac{dU(\theta_i'|\theta_i)}{d\theta_i'}\Big|_{\theta_i'=\theta_i} = 0$, which gives the condition

$$\frac{dw^-(G(\theta_i) - G(\gamma(\theta_i)))}{d\theta_i}u(\theta_i, -T) + \frac{dw^+(G(\gamma(\theta_i)))}{d\theta_i}u(\theta_i, 0) = 0, \quad \forall \theta_i \in [\underline{\theta}, \overline{\theta}]. \tag{10}$$

When T is sufficiently large, the necessary condition Equation (10) is also the sufficient condition under which reporting one's own value is the unique maximizer of expected utility because if γ satisfies Equation (10), then

$$\frac{d^2U(\theta_i'|\theta_i)}{d\theta_i d\theta_i'} = \frac{dw^+(G(\gamma(\theta_i')))}{d\theta_i'}\left[\frac{-u(\theta_i', T)}{u(\theta_i', 0)}u_\theta(\theta_i, -T) + u_\theta(\theta_i, 0)\right] > 0, \quad \forall \theta_i, \theta_i' \tag{11}$$

as long as T is large enough that $u(\theta_i', T) < 0$.[2]

It remains to show that there exists a function $\gamma : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$ that satisfies Equation (10). As G is strictly increasing, this is equivalent to finding a function $H : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ where $H(\theta_i) = G(\gamma(\theta_i))$ satisfies (10) written as an ODE

$$H'(\theta) = \frac{w^-(G(\theta_i) - H(\theta_i))[-u(\theta_i, -T)]g(\theta_i)}{w^-(G(\theta_i) - H(\theta_i))[-u(\theta_i, -T)] + w^+(H(\theta_i))u(\theta_i, 0)}, \tag{12}$$

such that $H(\underline{\theta}) = 0$ and $0 \leq H(\theta_i) \leq G(\theta_i)$ for all θ_i. As u, G, w^+, and w^- are differentiable in θ_i, Picard–Lindelöf theorem implies that the problem has a solution (in fact a unique solution).

[1] I assume here that T is big enough that paying T for the object falls into the losses domain.

[2] Note that this implies that $\frac{dU(\theta_i'|\theta_i)}{d\theta_i'} = \frac{dU(\theta_i'|\theta_i')}{d\theta_i'} + \int_{\theta_i'}^{\theta_i} \frac{d^2U(\theta_i'|t)}{d\theta_i' d\theta_i}dt \begin{cases} > 0 & \forall \theta_i' < \theta_i, \\ = 0 & \theta_i' = \theta_i, \\ < 0 & \forall \theta_i' > \theta_i. \end{cases}$

Therefore, for any T-mechanism with sufficiently large T, there exists a (unique) function γ with which truth-telling is optimal for each buyer, assuming that other buyers bid their types.

$\square$

Lemma 5. *If* $\lim_{T\to\infty} w^{-}(\frac{c}{T})u(\theta^*, T)$ *for each* $c > 0$*, then using* T*-mechanisms, the seller can ensure*

1. *unboundedly large expected transfers from each type who receives the object with positive probability,*
2. *unbounded expected profits.*

Proof I need to show that the expected transfer can be made arbitrarily large from all types $\theta_i \geq \theta^*$. The expected transfer from type θ_i, denoted by $t(\theta_i)$, is

$$t(\theta_i) = \int_{\gamma(\theta_i)}^{\theta_i} T dG(\theta_{-i}) = T[G(\theta_i) - G(\gamma(\theta_i))], \tag{13}$$

where the function γ is characterized by Equation (10) in the proof of Lemma 4. Suppose that the claim does not hold for some type θ_i; that is, $\lim_{T\to\infty} t(\theta_i) = c < \infty$. Then $\lim_{T\to\infty} \left[G(\theta_i) - G(\gamma(\theta_i)) - \frac{c}{T}\right] = 0$.

Notice that by assumptions, for any sufficiently large T,

$$u(\theta_i, 0) \geq u(\underline{\theta}, 0) > 0 > u(\theta_i, -T) \geq u(\underline{\theta}, -T). \tag{14}$$

Therefore, (10) implies that for all $\theta \geq \theta^*$,

$$\frac{dw^{-}(G(\theta_i) - G(\gamma(\theta_i)))}{d\theta_i}[-u(\underline{\theta}, -T)] \geq \frac{dw^{+}(G(\gamma(\theta_i)))}{d\theta_i}u(\underline{\theta}, 0). \tag{15}$$

Integrating both sides from $\underline{\theta}$ to θ_i and using the fact that $G(\underline{\theta}) = G(\gamma(\underline{\theta})) = 0$ gives

$$[-u(\underline{\theta}, -T)]w^{-}(G(\theta_i) - G(\gamma(\theta_i))) \geq u(\underline{\theta}, 0)w^{+}(G(\gamma(\theta_i))). \tag{16}$$

Note that $\lim_{T\to\infty} u(\underline{\theta}, 0)w^{+}(G(\gamma(\theta_i))) > 0$ because $u(\underline{\theta}, 0) > 0$ and $\lim_{T\to\infty} G(\gamma(\theta_i)) = 0$, which would imply that $\lim_{T\to\infty} t(\theta_i) = \infty$. The limit of the left-hand side of (16) is

$$\lim_{T\to\infty} [-u(\underline{\theta}, -T)]w^{-}(G(\theta_i) - G(\gamma(\theta_i))) = -\lim_{T\to\infty} w^{-}\left(\frac{c}{T}\right)u(\underline{\theta}, -T) = 0 \tag{17}$$

as the agents are asymptotically risk-loving. This is a contradiction.

$\square$