The Limits of Commitment

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Abstract

We parameterize commitment in leader-follower games by letting the leader publicly choose her action set from a menu of options. We fully characterize for a large class of settings the set of equilibrium outcomes obtained when varying the degree of commitment that the leader has. We identify conditions under which giving more commitment power to the leader could end up making her worse off. Moreover, with partial commitment, the follower might obtain a larger payoff than the leader even in settings where the latter possesses a first-mover advantage under full commitment. We explore the implications of our analysis for oligopolies.

JEL: C72, D43, D82

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1 Introduction

The Stackelberg model of commitment (von Stackelberg (1934)) has had a formidable impact on the development of industrial organization, political economy, international trade, and other areas of economic theory. A part of this impact is arguably explained by the sheer simplicity of the model, which, in turn, can be traced back to the leader's ability to commit exactly to the action of her choice. Yet this "full-commitment" assumption of the Stackelberg model lacks realism when, as often in applications, small adjustments to the leader's initial

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action are feasible. In oligopolistic contexts for instance (Spence (1977), Dixit (1980)), capacity investments could have multiple uses. For example, machines and employees can often be used in the production of more than just one good. The purpose of the present paper is (i) to provide a model enabling us to parameterize the *degree* of commitment in leader-follower environments, and (ii) to describe the set of outcomes thus obtained as a function of the base game considered (i.e., to elicit the "limits of commitment").[1]

The model we propose is simple. In the first period, the leader publicly selects a subset of actions from a given cover of her action space; this *exogenous* cover parameterizes the game considered. In the second period, leader and follower simultaneously choose an action, the leader having to pick an action from the cover element that she selected in the first period. We say that an outcome is *plausible* if it is a subgame perfect equilibrium outcome of the game induced by *some* cover of the leader's action space. By taking said cover to be the power set of the leader's action space, we effectively retrieve the Stackelberg model. When this cover only contains the action space itself, the first period becomes moot, and we then retrieve the simultaneous-move game associated with the base game considered.

To keep the analysis tractable, we restrict attention throughout to settings in which the action spaces can be represented by compact intervals of the real line. The paper's primary goal is to characterize the entire set of plausible outcomes, as well as three prominent subsets of plausible outcomes: the subset of outcomes that remain plausible under interval covers (i.e., covers made up of intervals only), the subset of outcomes which remain plausible under partitions, and the subset of outcomes which remain plausible under interval partitions.

A possible conjecture would be that the plausible actions of the leader coincide with the convex hull of her actions under full and no commitment, henceforth respectively referred to as "Stackelberg" and "Cournot" actions. As we show, however, generally the plausible actions of the leader are neither contained in the aforementioned convex hull nor do they contain it. We start by fully characterizing the sets of outcomes that are plausible under interval partitions and covers, respectively. In settings with a unique Cournot action, these sets take a very simple form: the actions of the leader which are plausible under interval partitions and covers both coincide with the upper contour set of the unique Cournot action with respect to the leader's indirect utility function.[2] With multiple Cournot actions, on the other hand, the aforementioned sets typically differ. For instance, an action of the leader is plausible under

[1] An "outcome" here means a pair of actions, one for the leader and one for the follower.

[2] The leader's indirect utility refers to the payoff of the leader given that the follower best-responds to the action which the leader takes.

interval covers if it is contained in the upper contour set of *any* Cournot action. However, for interval partitions the previous condition is generally insufficient: an action of the leader is plausible under interval partitions if it is contained in the upper contour sets of *all* Cournot actions. We then show that when the reaction curves are sufficiently steep, the set of plausible outcomes is strictly larger than the set of outcomes that are plausible under interval covers.

A first implication of our results is that the leader might prefer less commitment power rather than more. A second implication is that settings traditionally associated with a first-mover advantage (Gal-Or (1985)) might in fact exhibit a second-mover advantage under partial commitment. The following example illustrates both of these points in a textbook duopoly setting.[3]

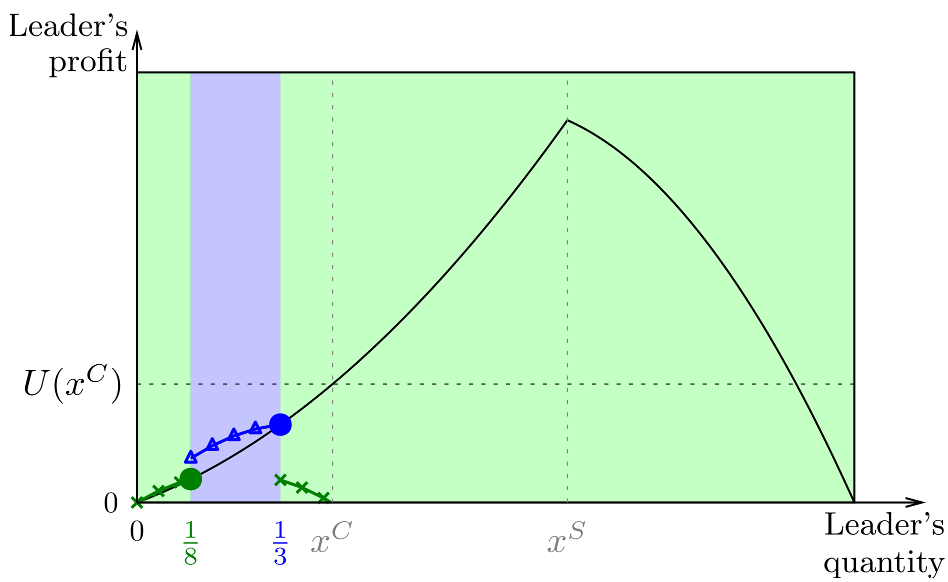

FIGURE 1: DUOPOLY EXAMPLE

Example. Leader and follower are two identical firms. In Figure 1, the quantity which the leader produces is represented on the horizontal axis: the leader's action space is given by $\mathcal{X} := \left[0, \frac{5}{3}\right]$. The black curve depicts the graph of the leader's indirect utility function (U). The Stackelberg and Cournot quantities are denoted by x^S and x^C, respectively. Next, let $\mathcal{I} := \left(\frac{1}{8}, \frac{1}{3}\right]$, and consider the cover K of the leader's action space given by $K := \left\{\mathcal{I}, \mathcal{X} \setminus \mathcal{I}\right\}$. The following strategies constitute a subgame perfect equilibrium of the game induced by K:

[3] In this example, the profit of the leader when leader and follower produce respectively x and y is given by $x(1-y) - \frac{3}{5}x^2$; the corresponding profit of the follower is $y(1-x) - \frac{3}{5}y^2$.

- on the equilibrium path, the leader commits to producing a quantity in $\mathcal{I}$, and subsequently chooses to produce $\frac{1}{3}$ (the blue curve indicates the leader's payoff conditional on the follower best-responding to $\frac{1}{3}$);

- off the equilibrium path, the leader produces $\frac{1}{8}$ (the green curve indicates the leader's payoff conditional on the follower best-responding to $\frac{1}{8}$);[4]

- the follower chooses to produce $\frac{5}{9}$ whenever the leader commits to $\mathcal{I}$, and $\frac{35}{48}$ whenever the leader commits to $\mathcal{X} \setminus \mathcal{I}$.

In this equilibrium the leader obtains a payoff of $U\left(\frac{1}{3}\right)$, which is less than $U(x^C)$ even though the cover K gives *some* commitment power to the leader. The follower, on the other hand, obtains a payoff that is greater than $U(x^C)$, and, therefore, greater than the payoff of the leader (albeit, under full commitment, the follower obtains a payoff that is less than $U(x^C)$, and, therefore, also less than the payoff of the leader in that case).

The final part of the paper studies the implications of our analysis for oligopolies. We first examine how competition and production technology affect in this case the limits of commitment. The more homogenous the products made, and the greater the returns to scale, the stronger the strategic motives of the leader, thus reinforcing the importance of commitment. Yet we show that competition and production technology have non-trivial effects on the plausible outcomes. For instance, making products more homogeneous might narrow the set of plausible actions of the leader. Finally, various optimal design problems are explored. We solve the problem of a designer aiming to maximize either the payoff of the leader, the payoff of the follower, consumer surplus, producer surplus, or total welfare. Plausible outcomes which involve the leader taking an action outside the convex hull of her Stackelberg and Cournot actions play a key role in this context. For instance, we show that any plausible outcome maximizing total welfare is such that the leader produces a quantity greater than her Stackelberg quantity.

The rest of the paper is organized as follows. The model and lead example are presented in Section 2. The general analysis is contained in Sections 3 and 4. Section 5 examines the implications of our analysis for oligopolies, and Section 6 concludes.

[4]That is, the leader produces $\frac{1}{8}$ whenever in the first period the leader commits to producing a quantity in the complement of $\mathcal{I}$.

Related Literature. We contribute to the literature on commitment, and, more specifically, to the strand of research exploring settings characterized by temporal asymmetries, in the vein of von Stackelberg (1934). We suggest a flexible yet tractable model to parameterize commitment in such settings. The model we propose is related to the model of Renou (2009); other related models include Saloner (1987), Admati and Perry (1991), and Romano and Yildirim (2005). In these papers, each player of a base game gradually restricts the set of actions that he will choose from in a terminal period; the environments considered and the kind of questions addressed are very different from ours, since players occupy temporally symmetric positions. Furthermore, all these papers fix the degree of commitment that the players have. For instance, in Renou (2009) each player has full commitment power, in the sense of being able to restrict his final action in any way that he desires.[5] The idea of partial commitment which lies at the heart of our study, connects our work to several classic papers. In Spence (1977) and Dixit (1980), the leader (i.e., the incumbent firm in that case) can pay a fraction of her production costs in advance. As this fraction goes from one to zero, the quantity produced by the leader spans exactly the range of values between the Stackelberg and Cournot quantities, nothing beyond that. Other prominent models of partial commitment include Maskin and Tirole (1988), Bagwell (1995), van Damme and Hurkens (1997), Maggi (1999), Henkel (2002), Várdy (2004), Caruana and Einav (2008), and Kamada and Kandori (2020).[6] The approach of the present paper is perhaps closest, in spirit at least, to Henkel (2002), Caruana and Einav (2008), and Kamada and Kandori (2020), where some players might have future opportunities to revise their choices, subject to switching costs or uncertainty.[7]

More broadly, our paper belongs to a recent literature taking a base game as given and exploring the set of outcomes resulting from allowing various aspects of the actual game to change. This literature includes Nishihara (1997), Kamenica and Gentzkow (2011), Bergemann, Brooks and Morris (2015), Bergemann and Morris (2016), Salcedo (2017), Taneva (2019), Makris and Renou (2021), Gallice and Monzón (2019), and Doval and Ely (2020)), among many others.

[5]In Renou (2009), it is the ability of *all* players to simultaneously commit to any subsets of actions which induces a rich and non-trivial set of equilibrium outcomes.

[6]In Maskin and Tirole (1988), commitment erodes over time. In Bagwell (1995), van Damme and Hurkens (1997), Maggi (1999), and Várdy (2004), imperfect monitoring hampers the leader's ability to commit.

[7]The focus of these papers is different from ours. Henkel (2002) studies strategic delays. Caruana and Einav (2008) and Kamada and Kandori (2020) seek to understand how revision opportunities affect cooperation.

2 Model and Lead Example

The model is described in Subsection 2.1. Subsection 2.2 introduces a number of key definitions and simplifying notation. The lead example used to illustrate our results throughout the paper is presented in Subsection 2.3.

2.1 The Model

There are two players, a *leader* and a *follower*, with action spaces $\mathcal{X} = [\underline{x}, \overline{x}]$ and $\mathcal{Y} = [\underline{y}, \overline{y}]$, respectively. The elements of $\mathcal{X} \times \mathcal{Y}$ are referred to as *outcomes*. We let $\mathcal{K}$ denote the set of covers of $\mathcal{X}$, that is, $K \in \mathcal{K}$ if and only if K is a collection of non-empty subsets of $\mathcal{X}$ whose union is equal to $\mathcal{X}$. The games that we consider comprise two stages. Given an exogenously fixed $K \in \mathcal{K}$:[8]

Stage 1. the leader publicly selects $\mathcal{X}_i \in K$;

Stage 2. leader and follower simultaneously choose actions x and y, with x contained in $\mathcal{X}_i$ and y contained in $\mathcal{Y}$.

An example in which K comprises three elements is depicted in Figure 2.

The payoffs are given by $u(x, y)$ for the leader and $v(y, x)$ for the follower, where u and v are twice continuously differentiable functions satisfying $u_{11} < 0$ and $v_{11} < 0$.[9] The game described above is denoted by $G(K)$. We refer to $G(2^{\mathcal{X}})$ as the *full-commitment game* and to $G(\{\mathcal{X}\})$ as the *no-commitment game*.

The central notion of our paper is:

Definition 1. *Let $\tilde{\mathcal{K}} \subseteq \mathcal{K}$. An outcome (x^*, y^*) is said to be* $\tilde{\mathcal{K}}$-plausible *if there exists a cover $K \in \tilde{\mathcal{K}}$ such that (x^*, y^*) is a subgame perfect equilibrium outcome of $G(K)$. An action $x^* \in \mathcal{X}$ is said to be* $\tilde{\mathcal{K}}$-plausible *if it forms part of a $\tilde{\mathcal{K}}$-plausible outcome.*

When we deem the chances of confusion sufficiently small, the term plausible will be used instead of $\mathcal{K}$-plausible.

[8]The model thus rules out any form of commitment to randomized devices.

[9]The assumption that u and v are differentiable is easily dispensed with, but simplifies the exposition a lot.

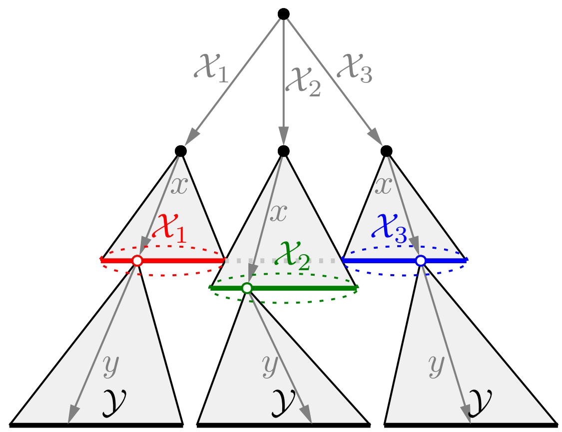

FIGURE 2: GAME TREE OF $G(K)$, WHERE $K = \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$

2.2 Additional Definitions and Notation

Three families of covers will be at the center of our attention: the collection of interval covers (i.e., covers comprising only intervals) is denoted by $\mathcal{K}^I$; the collection of partitions of $\mathcal{X}$ is denoted by $\mathcal{P}$; finally, the collection of interval partitions is denoted by $\mathcal{P}^I$, that is, $\mathcal{P}^I = \mathcal{P} \cap \mathcal{K}^I$.

The leader's action space being compact and u_{11} negative, to every $y \in \mathcal{Y}$ corresponds a unique best response $R_L(y)$ of the leader; similarly, as $\mathcal{Y}$ is compact and v_{11} negative, to every $x \in \mathcal{X}$ corresponds a unique best response $R_F(x)$ of the follower. Let

$$\phi(x) := R_L\big(R_F(x)\big).$$

The fixed points of ϕ are actions of the leader in (subgame perfect) Nash equilibria of the no-commitment game; such actions and Nash equilibria are referred to as *Cournot* actions and outcomes, respectively. The set of Cournot actions is denoted by $\mathcal{X}^C$, with x_n^C denoting a generic element of this set.

We let $U(x)$ be the payoff of the leader from taking action x when the follower best-responds to x, that is,

$$U(x) := u\big(x, R_F(x)\big).$$

The maximizers of U are actions of the leader in subgame perfect equilibria of the full-commitment game. Such actions are referred to as *Stackelberg* actions; the outcomes of these equilibria are called *Stackelberg* outcomes.

The upper contour set of x with respect to U is written $\mathcal{Q}_{\geq}(x)$, that is,

$$\mathcal{Q}_{\geq}(x) := \big\{\tilde{x} :\ U(\tilde{x}) \geq U(x)\big\}.$$

The sets $\mathcal{Q}_{>}(x)$, $\mathcal{Q}_{\leq}(x)$, and $\mathcal{Q}_{<}(x)$ are similarly defined. Whenever we deem the chances of confusion sufficiently small, we will talk about, e.g., the upper contour set of x, without explicit reference to U.

Let

$$\eta(\tilde{x}, x) := u\big(\tilde{x}, R_F(x)\big) - u\big(x, R_F(x)\big).$$

In words, $\eta(\tilde{x}, x)$ measures the leader's gain from deviating from x to $\tilde{x}$ when the follower best-responds to x.

Definition 2. *A pair (K, β) with $K \in \mathcal{K}$ and $\beta : K \to \mathcal{X}$ is said to be* admissible *if*

(a) $\beta(\mathcal{X}_i) \in \mathcal{X}_i$, *for all* $\mathcal{X}_i \in K$;

(b) $\eta\big(x, \beta(\mathcal{X}_i)\big) \leq 0$, *for all* $\mathcal{X}_i \in K$ *and all* $x \in \mathcal{X}_i$.

Note that a given cover may form part of several admissible pairs, or none at all. Furthermore, each admissible pair (K, β) is associated with at least one subgame perfect equilibrium of $G(K)$, and vice versa, as recorded by the following remark:

Remark 1. *An outcome (x^*, y^*) is $\tilde{\mathcal{K}}$-plausible if and only if there exists an admissible pair (K, β) with $K \in \tilde{\mathcal{K}}$ and $\mathcal{X}_{i^*} \in K$, such that*

(i) $x^* = \beta(\mathcal{X}_{i^*})$,

(ii) $U\big(\beta(\mathcal{X}_{i^*})\big) = \max_{\mathcal{X}_i \in K} U\big(\beta(\mathcal{X}_i)\big)$,

(iii) $y^* = R_F(x^*)$.

We then say that (K, β) implements outcome (x^*, y^*) (or action x^*).

2.3 The Duopoly Example

We formally introduce here our lead example. Leader and follower are two identical firms, each choosing a quantity in $\mathcal{X} = \mathcal{Y} = \big[0, \frac{2}{2-r}\big]$.[10] A firm producing quantity q incurs cost $3q - \frac{r}{2}q^2$

[10]Quantities larger than $2/(2-r)$ would lead to negative profits no matter what.

and sells at unit price $4 - (1-d)Q - dq$, where Q represents the total quantity produced by the two firms. In the previous expressions, $r < 2$ measures the returns to scale and $d \in [0,1]$ the degree of product differentiation. Letting $u(x,y)$ (respectively, $v(y,x)$) be the profit of the leader (respectively, the follower) when leader and follower respectively produce x and y gives $v(y,x) = u(y,x)$ and

$$u(x,y) = x - (1-d)xy - \Big(1 - \frac{r}{2}\Big)x^2. \tag{1}$$

3 Interval Plausibility

This section examines which outcomes are plausible under interval covers. Subsection 3.1 briefly explains why the set of $\mathcal{K}^I$-plausible outcomes generally differs from the set of $\mathcal{P}^I$-plausible ones; Subsection 3.2 provides characterizations of these two sets. A couple of important special cases are investigated in Subsection 3.3.

3.1 Preliminaries

As far as interval covers are concerned, the criteria for admissibility happen to take a very simple form, which we describe in the next lemma.

Lemma 1. *For $K \in \mathcal{K}^I$, the pair (K, β) is admissible if and only if, for all $\mathcal{X}_i \in K$, one of the following conditions holds:*

(i) $\beta(\mathcal{X}_i) \in \mathcal{X}_i \cap \mathcal{X}^C$;

(ii) $\beta(\mathcal{X}_i) = \min \mathcal{X}_i$ *and* $\phi\big(\beta(\mathcal{X}_i)\big) < \beta(\mathcal{X}_i)$;

(iii) $\beta(\mathcal{X}_i) = \max \mathcal{X}_i$ *and* $\phi\big(\beta(\mathcal{X}_i)\big) > \beta(\mathcal{X}_i)$.

The proof of the lemma is elementary; we relegate it to Appendix A. Figure 3, panel A, illustrates the result in the context of the duopoly example, for parameter values $d = 0$ and $r = 6/5$. The black curve represents the graph of the function ϕ; this curve crosses the 45-degree line at the Cournot actions $x_1^C = 0$, $x_2^C = 5/9$, and $x_3^C = 5/4$. An admissible pair (K, β) must be such that every action $\beta(\mathcal{X}_i)$ which belongs to a region of the figure comprising a left-pointing arrow (respectively, right-pointing arrow) is either a Cournot action, or the leftmost (respectively, rightmost) element of $\mathcal{X}_i$.

Lemma 1 enables us to make the following observation:

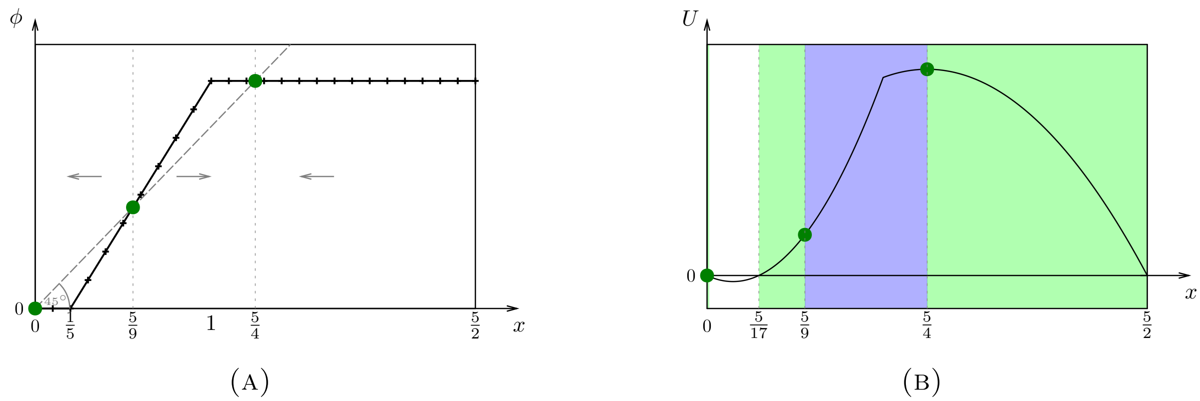

FIGURE 3: DUOPOLY EXAMPLE, FOR $d = 0$ AND $r = 6/5$

Remark 2. *If (K, β) is an admissible pair and $K \in \mathcal{P}^I$, then $\beta(K) \cap \mathcal{X}^C \neq \varnothing$.*[11]

The proof of this remark for the case in which the number of Cournot actions is finite is as follows.[12] Let $\mathcal{X}_0$ denote the partition element containing $\underline{x}$. Applying Lemma 1 shows that $\phi\big(\beta(\mathcal{X}_0)\big) \geq \beta(\mathcal{X}_0)$. If the previous inequality is an equality, then $\beta(\mathcal{X}_0) \in \mathcal{X}^C$; otherwise, let x_1^C denote the smallest Cournot action greater than $\beta(\mathcal{X}_0)$, and $\mathcal{X}_1$ the partition element containing x_1^C. As ϕ is continuous, we have $\phi(x) > x$ for all $x \in \big(\beta(\mathcal{X}_0), x_1^C\big)$.[13] So using Lemma 1 gives $\phi\big(\beta(\mathcal{X}_1)\big) \geq \beta(\mathcal{X}_1)$. The number of Cournot actions being finite, and using the fact that $\phi(\overline{x}) \leq \overline{x}$, we see by induction that $\beta(\mathcal{X}_i) \in \mathcal{X}^C$ for some $\mathcal{X}_i \in K$.

Interval covers are more permissive than interval partitions precisely because under interval covers all Cournot actions may be "bypassed" by β. To illustrate this point, return to the example of Figure 3. Now pick arbitrary actions x' and x'' such that $0 < x' < \frac{5}{9} < x'' < \frac{5}{4}$ and consider the pair (K, β) where $K = \big\{[0, x''], [x', \frac{5}{2}]\big\}$, $\beta\big([0, x'']\big) = x''$, and $\beta\big([x', \frac{5}{2}]\big) = x'$. Applying Lemma 1 shows that (K, β) is admissible; yet, here $\beta(K) \cap \mathcal{X}^C = \varnothing$. The ability to bypass Cournot actions often enlarges the set of $\mathcal{K}^I$-plausible outcomes relative to the set of $\mathcal{P}^I$-plausible ones. For instance, as we shall see, in the context of the previous example no action in $\big(\frac{5}{9}, \frac{5}{4}\big)$ is $\mathcal{P}^I$-plausible, though all of them are $\mathcal{K}^I$-plausible.

[11] Where $\beta(K)$ denotes the image of K under the mapping β.

[12] The proof is similar for the remaining case.

[13] By Berge's maximum theorem, both R_F and R_L are continuous, thus ϕ is continuous as well.

3.2 Main Results

We next proceed to characterize first the $\mathcal{P}^I$-plausible outcomes (Theorem 1), then the $\mathcal{K}^I$-plausible ones (Theorem 2).

Theorem 1. *An action x^* is $\mathcal{P}^I$-plausible if and only if there exists a Cournot action $x^C_{n^*}$ in the lower contour set of x^* with respect to U such that*

$$\big(\phi(x^*) - x^*\big)\big(x^C_{n^*} - x^*\big) \geq 0. \tag{2}$$

The proof of the *if* part of the theorem is easy. Consider x^* such that (2) holds for some $x^C_{n^*} \in \mathcal{Q}_{\leq}(x^*)$. All Cournot actions being trivially $\mathcal{P}^I$-plausible, suppose $\phi(x^*) > x^*$ (the remaining case is analogous). In this case, by (2), $x^C_{n^*} > x^*$. Now consider $K = \big\{[\underline{x}, x^*], (x^*, \overline{x}]\big\}$, and $\beta : K \to \mathcal{X}$ given by $\beta\big([\underline{x}, x^*]\big) = x^*$ and $\beta\big((x^*, \overline{x}]\big) = x^C_{n^*}$. By Lemma 1, the pair (K, β) is admissible. So (K, β) implements x^*, since $x^C_{n^*} \in \mathcal{Q}_{\leq}(x^*)$. The proof of the *only if* part of the theorem is in Appendix A, and rests on the fact that if a pair (K, β) with $K \in \mathcal{P}^I$ implements an action x^* such that, say, $\phi(x^*) > x^*$, then $\beta(\mathcal{X}_i) \in \mathcal{X}^C \cap (x^*, \overline{x}]$ for some $\mathcal{X}_i \in K$.

Applying Theorem 1 to the example of Figure 3 shows that the set of $\mathcal{P}^I$-plausible actions is $\{0\} \cup \big[\frac{5}{17}, \frac{5}{9}\big] \cup \big[\frac{5}{4}, \frac{5}{2}\big]$. Firstly, Theorem 1 shows that no action in the interval $\big(0, \frac{5}{17}\big)$ is $\mathcal{P}^I$-plausible, since all of them belong to the strict lower contour set of each Cournot action (see panel B). Secondly, any action $x \in \big(\frac{5}{9}, \frac{5}{4}\big)$ satisfies $\phi(x) > x$ (see panel A). The only Cournot action greater than any of these actions is $x^C_3 = 5/4$. As $U(x^C_3) > U(x)$ for all $x \in \big(\frac{5}{9}, \frac{5}{4}\big)$, we conclude using Theorem 1 that no action in this interval is $\mathcal{P}^I$-plausible. Mirror arguments show that all actions in $\{0\} \cup \big[\frac{5}{17}, \frac{5}{9}\big] \cup \big[\frac{5}{4}, \frac{5}{2}\big]$ are $\mathcal{P}^I$-plausible.[14]

The following corollary of Theorem 1 can oftentimes readily establish whether an action is $\mathcal{P}^I$-plausible or not.

Corollary 1. *Every action which belongs to the intersection of the upper contour sets of the Cournot actions with respect to U is $\mathcal{P}^I$-plausible. An action that does not belong to any of these upper contour sets is not $\mathcal{P}^I$-plausible.*

We next turn our attention to the collection of $\mathcal{K}^I$-plausible outcomes.

[14]An action $x^* \in [\frac{5}{17}, \frac{5}{9}] \cup [\frac{5}{4}, \frac{5}{2}]$ is for instance implemented by the pair (K, β) where $K = \big\{[0, x^*), [x^*, \frac{5}{2}]\big\}$, $\beta\big([0, x^*)\big) = 0$, and $\beta\big([x^*, \frac{5}{2}]\big) = x^*$.

Theorem 2. *An action x^* is $\mathcal{K}^I$-plausible if and only if both $\mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \leq x\}$ and $\mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\}$ are non-empty and*

$$\min \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \leq x\} \leq \max \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\}. \tag{3}$$

The proof of the *if* part of the theorem is straightforward. Suppose that both $\mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \leq x\}$ and $\mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\}$ are non-empty, and that (3) holds. Let x' and x'' be such that:

(i) $x' \in \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \leq x\}$,

(ii) $x'' \in \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\}$,

(iii) $x' \leq x''$.

Consider $K = \{\{x^*\}, [\underline{x}, x''], [x', \overline{x}]\}$, and $\beta : K \to \mathcal{X}$ given by $\beta(\{x^*\}) = x^*$, $\beta([\underline{x}, x'']) = x''$, and $\beta([x', \overline{x}]) = x'$. Applying Lemma 1 shows that (K, β) is admissible; so (K, β) implements x^*, since x' and x'' both belong to the lower contour set of x^*. The proof of the *only if* part of the theorem is in Appendix A, and rests on the observation that, given any admissible pair (K, β) with $K \in \mathcal{K}^I$, the image of β must contain actions x' and x'' which satisfy the properties (i)–(iii) previously listed.[15]

The following corollary of Theorem 2 is immediate.

Corollary 2. *Any action that belongs to the union of the upper contour sets of the Cournot actions with respect to U is $\mathcal{K}^I$-plausible.*

Applying Theorem 2 to the example in Figure 3 shows that the set of $\mathcal{K}^I$-plausible actions is $\{0\} \cup \left[\frac{5}{17}, \frac{5}{2}\right]$.[16] By Corollary 2, all actions in $\{0\} \cup \left[\frac{5}{17}, \frac{5}{2}\right]$ are $\mathcal{K}^I$-plausible. Moreover, any $x^* \in \left(0, \frac{5}{17}\right)$ is such that $\mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\} = \varnothing$. We conclude using Theorem 2 that no action in $\left(0, \frac{5}{17}\right)$ is $\mathcal{K}^I$-plausible.

Observe that in the previous example the unique Stackelberg action corresponds to $x_3^C = 5/4$, whence the convex hull of the Stackelberg and Cournot actions equals $\left[0, \frac{5}{4}\right]$. It follows that the set of $\mathcal{K}^I$-plausible actions is neither contained in nor contains the convex hull of the Stackelberg and Cournot actions.[17]

[15] Notice that if we assumed instead that $K \in \mathcal{P}^I$, we would in addition require that $x' = x''$ (Remark 2).

[16] Actions in $\left(\frac{5}{9}, \frac{5}{4}\right)$ are $\mathcal{K}^I$-plausible, but are not $\mathcal{P}^I$-plausible. An action x^* in this interval is for instance implemented by the pair (K, β) where $K = \{[0, \frac{5}{2}], [0, x^*]\}$, $\beta([0, \frac{5}{2}]) = 0$, and $\beta([0, x^*]) = x^*$.

[17] Furthermore, since here the leader can ensure a payoff of 0 by choosing $x = 0$, we see that no action in

3.3 Further Analysis

Corollary 2 raises the following question: does the set of $\mathcal{K}^I$-plausible actions generally coincide with the union of the upper contour sets of the Cournot actions? The following example shows that it needs not.

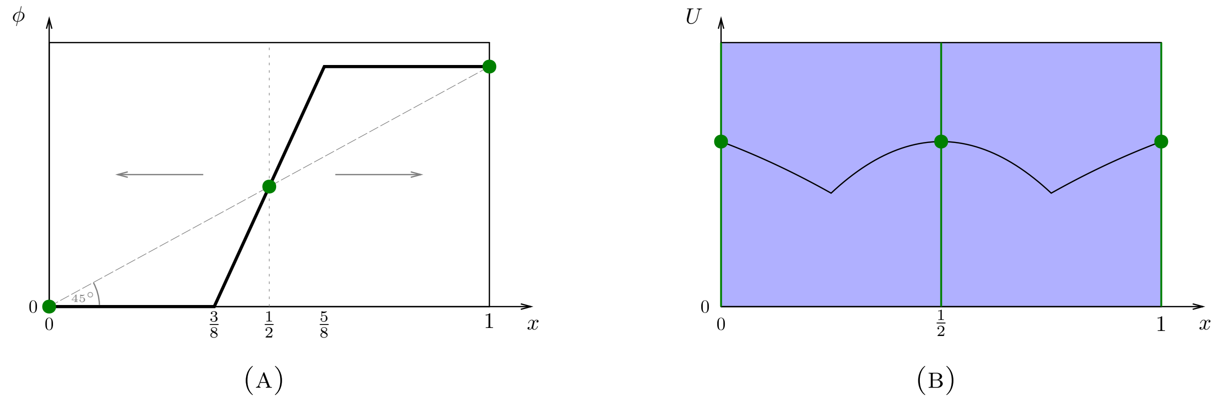

Figure 4: Example

Example. Let $\mathcal{X} = \mathcal{Y} = [0, 1]$,

$$u(x, y) = xy + (1 - x)(1 - y) - \frac{(x - \frac{1}{2})^2}{2} - \frac{3(y - \frac{1}{2})^2}{2},$$

and $v(y, x) = u(y, x)$. Pick an arbitrary action $x^* \geq 1/2$, and consider the pair (K, β) in which $K = \big\{[0, x^*], [1-x^*, 1]\big\}$ and $\beta\big([0, x^*]\big) = x^* = 1-\beta\big([1-x^*, 1]\big)$. Figure 4, panel A, shows that $\phi(x^*) \geq x^*$ and $\phi(1-x^*) \leq 1-x^*$; on the other hand, panel B shows that $U(x^*) = U(1-x^*)$. We conclude using Lemma 1 that (K, β) constitutes an admissible pair, which implements x^* as well as $1 - x^*$. It follows that all actions in $[0, 1]$ are $\mathcal{K}^I$-plausible. Yet the union of the upper contour sets of the Cournot actions contains no other action than the Cournot actions themselves.

The previous example gives an early illustration of the fact that the welfare of the leader need not be monotonic in the degree of commitment afforded to the latter. Here, the payoff which the leader obtains equals 1/2 both in the full- and no-commitment games, yet inter-

$(0, \frac{5}{17})$ is $\mathcal{K}$-plausible. The remark in the text therefore applies to the $\mathcal{K}$-plausible actions too: neither are they contained in the convex hull of the Stackelberg and Cournot actions, nor do they contain it.

mediate degrees of commitment can leave the leader with a payoff as low as 11/32. We will return to this important insight in the next section.

As a counterpoint to the example above, the following proposition shows that in somewhat well-behaved environments, the set of $\mathcal{K}^I$-plausible actions in fact coincides with the union of the upper contour sets of the Cournot actions.

Proposition 1. *Suppose U is either quasi-convex or quasi-concave. An action is $\mathcal{K}^I$-plausible if and only if it belongs to the union of the upper contour sets of the Cournot actions with respect to U.*

Proof: We already know by Corollary 2 that an action that belongs to the upper contour set of some Cournot action is $\mathcal{K}^I$-plausible. Below we show that the converse is true too if U is either quasi-convex or quasi-concave.

Suppose that U is quasi-convex, and consider an action x^* in the strict lower contour set of every Cournot action. Then $\mathcal{Q}_{\leq}(x^*)$ is a convex set, and $\phi(x) \neq x$ for all $x \in \mathcal{Q}_{\leq}(x^*)$. The intermediate value theorem shows that either $x < \phi(x)$ for all $x \in \mathcal{Q}_{\leq}(x^*)$, or $x > \phi(x)$ for all $x \in \mathcal{Q}_{\leq}(x^*)$. Either way, Theorem 2 shows that x^* cannot be $\mathcal{K}^I$-plausible.

Next, suppose that U is quasi-concave, and consider an action x^* in the strict lower contour set of every Cournot action. Then $\mathcal{Q}_{>}(x^*)$ is a convex set, and $\phi(x) \neq x$ for all $x \in \mathcal{Q}_{\leq}(x^*)$. This implies that, given $x \in \mathcal{Q}_{\leq}(x^*)$, either (i) $\phi(x) > x$ and $x < x_n^C$ for all $x_n^C \in \mathcal{X}^C$, or (ii) $\phi(x) < x$ and $x > x_n^C$ for all $x_n^C \in \mathcal{X}^C$. We conclude using Theorem 2 that x^* is not $\mathcal{K}^I$-plausible. ■

Finally, combining Theorems 1 and 2 yields:[18]

Proposition 2. *If there exists a unique Cournot outcome, then the set of $\mathcal{K}^I$-plausible actions is also the set of $\mathcal{P}^I$-plausible ones, and this set coincides with the upper contour set of the unique Cournot action with respect to U.*

Proof: Suppose that there exists a unique Cournot action; denote it by x^C. Applying Corollary 1 shows that every $x^* \in \mathcal{Q}_{\geq}(x^C)$ is $\mathcal{P}^I$-plausible. Next, observe that $\{x : \phi(x) \geq x\} = [\underline{x}, x^C]$ and $\{x : \phi(x) \leq x\} = [x^C, \overline{x}]$. Applying Theorem 2 thus shows that if x^* is $\mathcal{K}^I$-plausible, x^C must belong to the lower contour set of x^*. ■

[18] A self-contained proof of Proposition 2 is provided in Appendix B. We rely here on this section's theorems.

4 Beyond Interval Plausibility

In this section, we extend our investigation beyond interval covers of the leader's action space. Subsection 4.1 characterizes the set of $\mathcal{K}$-plausible outcomes for a class of settings comprising a unique Cournot outcome. In Subsection 4.2, we show that when the reaction curves are sufficiently steep the $\mathcal{K}$-plausible outcomes form a strict superset of the $\mathcal{K}^I$-plausible ones, and characterize the minimal subset of covers generating all plausible outcomes.

4.1 Main Result

To keep the analysis tractable, we focus in this section on settings satisfying the following three regularity conditions:

(RC1) $\mathcal{X}^C = \{x^C\}$, with $x^C \in \text{int}(\mathcal{X})$ and $y^C := R_F(x^C) \in \text{int}(\mathcal{Y})$;

(RC2) $u_2 v_2 > 0$;

(RC3) $u_{12} v_{12} > 0$.

Condition (RC1) supposes the existence of a unique Cournot outcome, (x^C, y^C). Condition (RC2) ensures homogenous payoff externalities: these could be positive or negative, but cannot change sign. Similarly, condition (RC3) ensures homogenous strategic interactions: actions may be strategic complements or substitutes, but cannot be both.

For every $x \in \mathcal{X}$, the function $\eta(\cdot, x)$ is strictly concave and satisfies $\eta(x, x) = 0$. It follows that $\eta(\tilde{x}, x) = 0$ for at most one action $\tilde{x}$ different from x. We can thus define $\gamma : \mathcal{X} \to \mathcal{X}$ as follows:

- if $\eta(\tilde{x}, x) = 0$ for some $\tilde{x} \neq x$, set $\gamma(x) = \tilde{x}$;
- otherwise, set

$$\gamma(x) = \begin{cases} \overline{x} & \text{if } x < x^C, \\ x^C & \text{if } x = x^C, \\ \underline{x} & \text{if } x > x^C. \end{cases}$$

The interpretation is straightforward: in cases where such an action exists, $\gamma(x)$ is the action making the leader indifferent between choosing x or $\gamma(x)$ when the follower best-responds to x.

Lastly, let

$$\mathcal{S} := \begin{cases} \{x : \; x \le \gamma(x) \le x^C\} & \text{if } u_2 u_{12} > 0, \\ \{x : \; x^C \le \gamma(x) \le x\} & \text{if } u_2 u_{12} < 0. \end{cases}$$

As γ is continuous, $\mathcal{S}$ is compact.[19] Moreover, $\mathcal{S}$ evidently contains x^C. We are now ready to state this section's main result.

Theorem 3. *Suppose (RC1)–(RC3) hold. The set of $\mathcal{K}$-plausible actions is also the set of $\mathcal{P}$-plausible actions; this set coincides with the upper level set of $\underline{U} := \min_{x \in \mathcal{S}} U\big(\gamma(x)\big)$ with respect to U.*[20]

The proof of Theorem 3 is in Appendix B. Return to the duopoly example, with parameter values $d = 0$ and $r = 4/5$. In Figure 5, panel A, the black curve represents the graph of the function ϕ, which crosses the 45-degree line at $x^C = 5/11$. Notice that in this example $u_2 u_{12} > 0$, so $\mathcal{S} = \{x : \; x \le \gamma(x) \le x^C\}$. The gray curve represents the graph of γ: we see that $\mathcal{S} = [0, x^C]$ and $\gamma(\mathcal{S}) = \left[\frac{5}{18}, x^C\right]$. Panel B depicts the graph of the function U. Minimizing U over $\gamma(\mathcal{S})$ shows that $\underline{U} = U\left(\frac{5}{18}\right)$. The upper level set of $\underline{U}$ corresponds to $\left[\frac{5}{18}, \hat{x}_2\right]$. The combination of Theorem 3 and Proposition 2 shows that the set of $\mathcal{K}$-plausible outcomes is strictly larger than the set of $\mathcal{K}^I$-plausible ones, since here the upper contour set of x^C is $\left[x^C, \hat{x}_1\right]$.

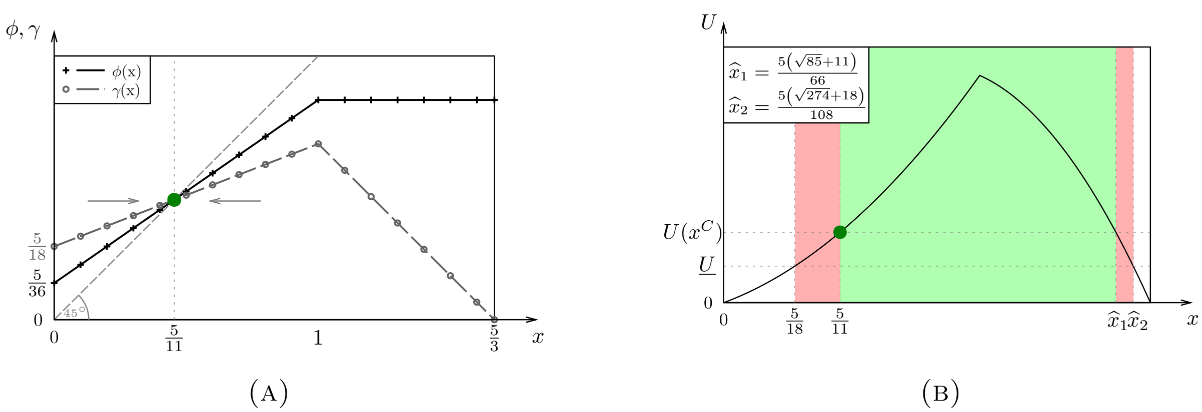

FIGURE 5: DUOPOLY EXAMPLE, FOR $d = 0$ AND $r = 4/5$

To partly illustrate the workings of Theorem 3, consider in the previous example $x^* = 1/3$. As $x^* \notin \mathcal{Q}_{\ge}(x^C) = \left[x^C, \hat{x}_1\right]$ (see panel B), the action x^* is not $\mathcal{K}^I$-plausible. Now let

[19] The continuity of γ is inherited from the continuity of u and R_F.

[20] The upper level set of $\underline{U}$ with respect to U is defined as $\{x : U(x) \ge \underline{U}\}$.

$\mathcal{X}_1 := \{0\} \cup \left(\frac{1}{3}, \frac{5}{3}\right]$, $\mathcal{X}_2 := \left(0, \frac{1}{3}\right]$, and consider the pair (K, β) where $K = \{\mathcal{X}_1, \mathcal{X}_2\}$, $\beta(\mathcal{X}_1) = 0$, and $\beta(\mathcal{X}_2) = \frac{1}{3}$. The function $\eta\left(\cdot, \frac{1}{3}\right)$ is strictly concave, maximized at $\phi\left(\frac{1}{3}\right) > \frac{1}{3}$, and equal to 0 at $\frac{1}{3}$. Therefore, $\eta\left(x, \frac{1}{3}\right) < 0$ for all $x < \frac{1}{3}$. Next, as $\phi(0) > 0$ the definition of γ yields $\eta(x, 0) < 0$ for every $x > \gamma(0)$. Since $\gamma(0) < \frac{1}{3}$ (see panel A), we thus obtain $\eta(x, 0) < 0$ for all $x \in \left(\frac{1}{3}, \frac{5}{3}\right]$. Combining the previous observations shows that (K, β) constitutes an admissible pair. Moreover, as $U(x^*) > U(0)$, we see that (K, β) implements x^*.

The previous paragraph illustrates once more that the leader could be better off with less commitment rather than more. Moreover, as here $v_2 < 0$, the fact that $x^* < x^C$ implies that the corresponding equilibrium payoff of the follower is larger than $U(x^C)$. In sum, although $G(2^{\mathcal{X}})$ is a game with a first-mover advantage (Gal-Or (1985)), $G(K)$ here results in a second-mover advantage.

4.2 Further Analysis

This subsection addresses two questions: (i) What are the general conditions under which the set of $\mathcal{K}$-plausible outcomes is strictly larger than the set of $\mathcal{K}^I$-plausible ones? (ii) What is the minimal subset of covers generating all plausible outcomes?

We show in Appendix B that the answer to question (i) is determined by the shape of γ: whenever $u_2 u_{12} > 0$ (respectively, $u_2 u_{12} < 0$) the $\mathcal{K}$-plausible outcomes form a strict superset of the $\mathcal{K}^I$-plausible ones if and only if $\gamma(x) < x^C$ for some $x < x^C$ (respectively, $\gamma(x) > x^C$ for some $x > x^C$). A simple sufficient condition for the latter condition to hold is plainly that $\gamma'(x^C) > 0$. Calculations relegated to Appendix B establish that $\gamma'(x^C) > 0$ if and only if $R'_L(y^C) R'_F(x^C) > 1/2$. We thus obtain:

Proposition 3. *Suppose (RC1)–(RC3) hold. If $R'_L(y^C) R'_F(x^C) > 1/2$ the $\mathcal{K}$-plausible outcomes then form a strict superset of the $\mathcal{K}^I$-plausible outcomes.*

Below, let $\mathcal{P}^{I+}$ denote the collection of partitions of $\mathcal{X}$ such that at most one element of the partition considered is not an interval. The answer to question (ii) is provided by the next result, whose proof is in Appendix B.

Proposition 4. *Suppose (RC1)–(RC3) hold. The set of $\mathcal{K}$-plausible outcomes is also the set of $\mathcal{P}^{I+}$-plausible outcomes.*

In fact, it can be shown that if U is either quasi-concave or quasi-convex then all plausible outcomes can be implemented with a partition of the leader's action space such that each partition element is either an interval or a union of two intervals. In such environments, very simple commitments generate the entire set of plausible outcomes: effectively, the leader is plainly asked to commit to choosing an action: (a) inside or outside an interval, (b) below or above a cutoff. This remark is particularly useful when taking the perspective of a designer controlling the leader's commitment. Several design problems of this kind are examined in the next section.

5 Application: Duopoly

Up to this point, we used the duopoly example to illustrate the workings of the model and the results of our analysis. In this section, we instead use our results to shed light on the limits of commitment in duopolies: throughout this section, $u(x, y)$ is given by (1) and $v(y, x) = u(y, x)$.

5.1 The Limits of Commitment in Duopolies

How do competition and production technology respectively affect the limits of commitment in duopolies? The formal analysis forming the basis of the discussion which follows is relegated to Appendix C. The main points of this analysis are summarized in Figures 6 and 7.

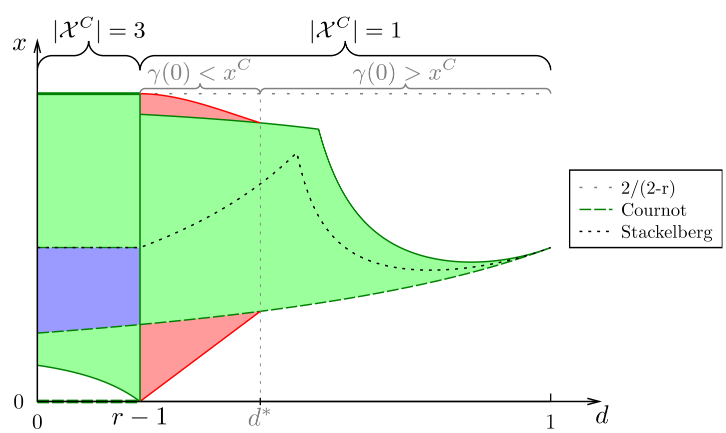

Figure 6: Comparative Statics with respect to d for $r = 6/5$

In Figure 6, the horizontal axis measures the degree of product differentiation: $d = 0$ corresponds to the homogenous products case, and $d = 1$ to the perfectly differentiated products case. In Figure 7, the horizontal axis measures returns to scale: decreasing for $r < 0$ and increasing for $r > 0$. In both figures, the vertical axis represents the leader's action space, $\left[0, \frac{2}{2-r}\right]$. The different colors identify the different subsets of plausible actions:

- an action is $\mathcal{P}^I$-plausible if and only if it is colored in green;
- an action is $\mathcal{K}^I$-plausible if and only if it is colored in green or blue;
- an action is $\mathcal{P}$-plausible if and only if it is colored in green, blue, or red;
- in this example the $\mathcal{K}$-plausible actions coincide with the $\mathcal{P}$-plausible ones.

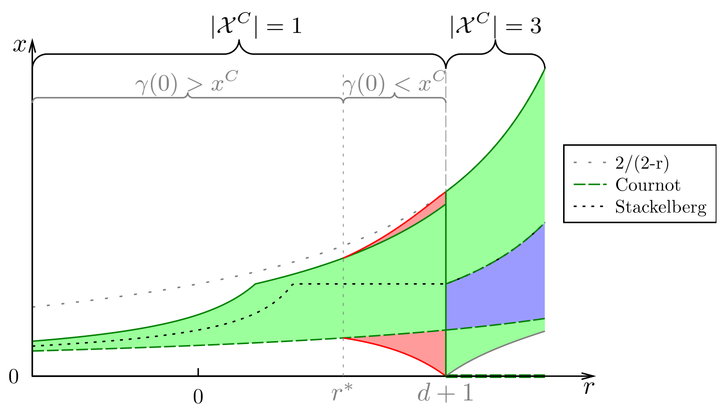

FIGURE 7: COMPARATIVE STATICS WITH RESPECT TO r FOR $d = 0$

Notice that, in both figures, a regime switch occurs where $d = r - 1$; at this cutoff, the number of Cournot actions changes:

- for $d < r - 1$, the set of Cournot actions is given by $\mathcal{X}^C = \{x_1^C, x_2^C, x_3^C\}$, where $x_1^C = 0$, $x_2^C = \frac{1}{3-r-d}$ and $x_3^C = \frac{1}{2-r}$;
- for $d > r - 1$, the unique Cournot action is $x^C = \frac{1}{3-r-d}$.

We discuss below the mechanisms at play in these two regions of parameters.

For $d < r - 1$: As $x_1^C = 0$, using Corollary 2 establishes that all actions in the upper contour set of 0 are $\mathcal{K}^I$-plausible. Furthermore, producing 0 yields a payoff of 0 regardless of the quantity produced by the other firm. Combining the previous remarks shows that the sets of $\mathcal{K}^I$- and $\mathcal{K}$-plausible actions both coincide with the upper contour set of 0.[21] Next, note that:

(a) reducing the degree of competition increases U,

[21] In fact, the set of $\mathcal{P}$-plausible actions also coincides in this case with the upper contour set of 0. To see this, pick an arbitrary action $x^* \in \mathcal{Q}_{\geq}(0)$, and consider the pair (K, β) where $K := \{\{x^*\}, \mathcal{X} \setminus \{x^*\}\}$, $\beta(\{x^*\}) = x^*$, and $\beta(\mathcal{X} \setminus \{x^*\}) = 0$.

(b) increasing the returns to scale causes the graph of U to tilt anti-clockwise.

Consequently, the softer the competition, the larger the set of quantities giving the leader a non-negative payoff. On the other hand, the more increasing the returns to scale, the greater (respectively, smaller) the payoff of the leader from producing large (respectively, small) quantities. In Figure 6, the set of $\mathcal{K}$-plausible actions thus expands as d increases from 0 to $r-1$. By contrast, in Figure 7, the set of $\mathcal{K}$-plausible actions shifts upwards as r increases past $d+1$.

For $d > r-1$: We show in Appendix C that the threshold $\underline{U} = \min_{x \in \mathcal{S}} U\big(\gamma(x)\big)$ is equal to $U\big(\gamma(0)\big)$ as long as $\gamma(0)$ is less than the unique Cournot action and equal to $U(x^C)$ otherwise. Using Theorem 3 shows that the $\mathcal{K}$-plausible actions then coincide with the upper contour set of $\min\big\{\gamma(0), x^C\big\}$. As softening competition increases $U\big(\gamma(0)\big)$ as well as $U\big(x^C\big)$, the set of $\mathcal{K}$-plausible actions thus shrinks as d goes from $r-1$ to 1 (see Figure 6). By contrast, as r increases, the interval of $\mathcal{K}$-plausible actions:

(i) shrinks (respectively, expands) at its lower (respectively, upper) end in the region of Figure 7 where $\min\big\{\gamma(0), x^C\big\} = x^C$,[22]

(ii) expands at both of its ends in the region of Figure 7 where $\min\big\{\gamma(0), x^C\big\} = \gamma(0)$.[23]

Part (i) is explained as follows. In that region, the lower end of the interval of $\mathcal{K}$-plausible actions coincides with x^C, which increases as production technology improves. The upper end of the interval, on the other hand, coincides with the largest quantity of the leader giving her the same payoff as the unique Cournot action. Part (ii) follows from the fact that improving production technology reduces $U\big(\gamma(0)\big)$ by causing $\gamma(0)$ to fall.

5.2 Optimal Commitment in Duopolies

What is the optimal form of commitment from the perspective of a designer interested in maximizing, say, consumer surplus? To answer this question and other related questions, we analyze in this subsection problems of the form

$$\max W(x,y) \quad \text{s.t. } (x,y) \text{ is } \mathcal{K}\text{-plausible.} \tag{P}$$

[22] I.e., for $r < r^*(d) = 2 - \sqrt{2}(1-d)$).

[23] I.e., for $r^*(d) < r < d+1$.

Below, let $\underline{x}^{\mathcal{K}}$ (respectively, $\overline{x}^{\mathcal{K}}$) denote the smallest (respectively, largest) $\mathcal{K}$-plausible action. We first examine situations in which the designer is one of the two firms. Trivially, the Stackelberg outcome is the best possible $\mathcal{K}$-plausible outcome from the perspective of the leader. On the other hand, since v_2 is here negative, the optimal $\mathcal{K}$-plausible outcome from the perspective of the follower involves the leader producing as little as plausibly possible. The proposition which follows summarizes these observations.

Proposition 5.

(i) For $W = u$, the unique solution of (P) *is* $\big(x^S, R_F(x^S)\big)$.

(ii) For $W = v$, the unique solution of (P) *is* $\big(\underline{x}^{\mathcal{K}}, R_F(\underline{x}^{\mathcal{K}})\big)$.

By Proposition 5, full commitment is optimal for the leader. On the other hand, from the perspective of the follower, no commitment is optimal if and only if $r \notin \big(r^*(d), d+1\big)$. For $r \in \big(r^*(d), d+1\big)$, the smallest $\mathcal{K}$-plausible action is not a Cournot action. In this case, the simplest follower-optimal cover takes the form of

$$K = \Big\{\big(0, \gamma(0)\big], \{0\} \cup \big(\gamma(0), \overline{x}\big]\Big\},$$

that is, the leader either commits to producing a quantity in the interval $\big(0, \gamma(0)\big]$, or commits to producing a quantity outside of this interval.

We next examine situations in which the designer aims to maximize either consumer surplus, producer surplus, or total welfare (i.e., the sum of producer and consumer surplus). We follow Singh and Vives (1984) and define the consumer surplus generated by an outcome (x, y) by[24]

$$CS(x, y) = \frac{(x+y)^2}{2} - dxy.$$

Producer surplus is simply defined by

$$PS(x, y) = u(x, y) + v(y, x).$$

Proposition 6.

(i) For W equal to consumer surplus, the unique solution of (P) *is* $\big(\overline{x}^{\mathcal{K}}, R_F(\overline{x}^{\mathcal{K}})\big)$.

[24] The expression for consumer surplus is based on the representative consumer utility function, given by $4(x+y) - \frac{1}{2}(x+y)^2 + dxy$.

(ii) For W equal to producer surplus, the unique solution of (P) *is* $\big(x^C, R_F(x^C)\big)$ *if* $r < r^\dagger(d)$*, and* $\big(x^S, R_F(x^S)\big)$ *if* $r^\dagger(d) < r < d+1$;[25] *if* $r > d+1$ *then the solutions are* $\big(x_3^C, 0\big)$ *and* $\big(0, x_3^C\big)$.

(iii) For W equal to total welfare, the unique solution of (P) *is* $\big(\overline{x}^{\mathcal{K}}, R_F(\overline{x}^{\mathcal{K}})\big)$.

Part (i) of Proposition 6 is explained as follows. Firstly, we show that consumer surplus is a convex function of the quantity which the leader produces. The problem of the designer therefore reduces to choosing between $\big(\underline{x}^{\mathcal{K}}, R_F(\underline{x}^{\mathcal{K}})\big)$ and $\big(\overline{x}^{\mathcal{K}}, R_F(\overline{x}^{\mathcal{K}})\big)$. Inducing the leader to produce $\overline{x}^{\mathcal{K}}$ instead of $\underline{x}^{\mathcal{K}}$ is optimal because in this way the designer can exploit the strategic motive to produce large quantities which arises from commitment. With multiple Cournot actions, or if there exists a single Cournot action and $\gamma(0) \geq x^C$, the binary partition $\big\{[\underline{x}, \overline{x}^{\mathcal{K}}), [\overline{x}^{\mathcal{K}}, \overline{x}]\big\}$ is consumer-optimal. Otherwise, the simplest consumer-optimal cover takes the form of

$$K = \Big\{\big(0, \gamma(0)\big], \{0\} \cup \big(\gamma(0), \overline{x}^{\mathcal{K}}\big), \big[\overline{x}^{\mathcal{K}}, \overline{x}\big]\Big\},$$

that is, the leader either commits to producing a quantity in the interval $\big(0, \gamma(0)\big]$, or commits to producing a quantity outside of this interval; in the latter case, the leader either commits to producing a quantity at least as large as $\overline{x}^{\mathcal{K}}$, or commits to producing less than this.

Part (ii) of Proposition 6 is straightforward. With decreasing returns to scale, producer surplus is maximized by inducing both firms to produce the same quantity; in this case, no-commitment is producer-optimal. In contrast, with large returns to scale, producer surplus is maximized by letting one firm acquire a bigger market share than the other firm. In particular, for very large returns to scale, producer surplus is maximized by letting one firm act as a monopolist. Consequently, no-commitment is producer-optimal for extreme returns to scale, whereas full-commitment is producer-optimal for sufficiently large returns to scale.

Part (iii) of Proposition 6 follows from the fact that producer surplus tends to be less sensitive than consumer surplus to the quantity which the leader produces. So maximizing total welfare implies maximizing consumer surplus.

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[25] Where $r^\dagger(d) := 2 - \left(\frac{\sqrt[3]{3(9-\sqrt{78})}}{3} + \frac{1}{\sqrt[3]{3(9-\sqrt{78})}}\right)(1-d)$.

6 Conclusion

This paper proposes a model to parameterize commitment in leader-follower games. Said model enables us to relax the full-commitment assumption implicit in the formulation of the Stackelberg model. We thereby account for the observation that, in many economic applications of the Stackelberg model, partial adjustments to the leader's initial action are in fact feasible.

In settings where the action spaces can be represented by compact intervals of the real line, our model turns out to be remarkably tractable, allowing us to characterize the rich set of equilibrium outcomes that arise for different degrees of commitment that the leader might be endowed with. In doing so, our study uncovers novel insights. For instance, we identify conditions under which giving more commitment power to the leader could end up hurting her. We also show that settings traditionally associated with a first-mover advantage can result in a second-mover advantage due to the inability of the leader to commit exactly to the action of her choice. More than commitment, what matters, our paper highlights, is the precise *form* of commitment afforded to the leader.

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A Appendix of Section 3

Proof of Lemma 1: We prove the *only if* part of the lemma; the proof of the other part is similar. Suppose that (K, β) constitutes an admissible pair. Reason by contradiction, and suppose that we can find $\mathcal{X}_i \in K$ such that $\phi\big(\beta(\mathcal{X}_i)\big) < \beta(\mathcal{X}_i)$ while $\beta(\mathcal{X}_i) \neq \min \mathcal{X}_i$. The function $\eta\big(\cdot, \beta(\mathcal{X}_i)\big)$ is strictly concave, maximized at $\phi\big(\beta(\mathcal{X}_i)\big)$, and satisfies $\eta\big(\beta(\mathcal{X}_i), \beta(\mathcal{X}_i)\big) = 0$. So $\eta\big(x, \beta(\mathcal{X}_i)\big) > 0$ for all $x \in \big[\phi\big(\beta(\mathcal{X}_i)\big), \beta(\mathcal{X}_i)\big)$. Since $\mathcal{X}_i$ is an interval, $\beta(\mathcal{X}_i) \in \mathcal{X}_i$, and $\beta(\mathcal{X}_i) \neq \min \mathcal{X}_i$, we can find $\varepsilon > 0$ such that $\big(\beta(\mathcal{X}_i) - \varepsilon, \beta(\mathcal{X}_i)\big) \subset \mathcal{X}_i$. Coupling the previous remarks shows the existence of $x \in \mathcal{X}_i$ such that $\eta\big(x, \beta(\mathcal{X}_i)\big) > 0$; this contradicts the assumption that (K, β) is admissible. Hence, $\phi\big(\beta(\mathcal{X}_i)\big) < \beta(\mathcal{X}_i)$ implies $\beta(\mathcal{X}_i) = \min \mathcal{X}_i$. Analogous arguments show that $\phi\big(\beta(\mathcal{X}_i)\big) > \beta(\mathcal{X}_i)$ implies $\beta(\mathcal{X}_i) = \max \mathcal{X}_i$. ■

Proof of Theorem 1: The *if* part of the theorem was proven in the text; we prove here the converse. Pick an arbitrary $\mathcal{P}^I$-plausible action x^*. We aim to prove the existence of a Cournot action $x^C_{n^*} \in \mathcal{Q}_\leq(x^*)$ such that (2) holds. If $\phi(x^*) = x^*$, just take $x^C_{n^*} = x^*$; we treat below the case in which $\phi(x^*) > x^*$ (the remaining case is analogous). Reason by contradiction, and suppose that

$$\mathcal{X}^C \cap (x^*, \overline{x}] \cap \mathcal{Q}_\leq(x^*) = \varnothing. \tag{4}$$

Let (K, β) implement x^*, where $K \in \mathcal{P}^I$. We will show that K cannot be a finite cover. By Berge's maximum theorem, both R_F and R_L are continuous, thus ϕ is continuous as well. As $\phi(x^*) > x^*$ and $\phi(\overline{x}) \leq \overline{x}$, the intermediate value theorem shows that

$$\mathcal{X}^C \cap (x^*, \overline{x}] \neq \varnothing.$$

Note that the continuity of the function ϕ implies the compactness of $\mathcal{X}^C$. So $\mathcal{X}^C \cap (x^*, \overline{x}] = \mathcal{X}^C \cap [x^*, \overline{x}]$ possesses a smallest element, that we denote by x_1^C. Let $\mathcal{X}_1$ be the member of K containing x_1^C. Then Lemma 1 combined with (4) gives

$$\beta(\mathcal{X}_1) \in (x_1^C, \overline{x}] \cap \big\{x : \phi(x) > x\big\}.$$

Now let x_2^C be the smallest Cournot action greater than $\beta(\mathcal{X}_1)$, and denote by $\mathcal{X}_2$ the member of K containing x_2^C. The same logic as above gives $\beta(\mathcal{X}_2) \in (x_2^C, \overline{x}] \cap \big\{x : \phi(x) > x\big\}$, and so on. If K were finite, the previous iteration would have to end after finitely many steps, say m. But then $\beta(\mathcal{X}_m) = \overline{x}$ and $\beta(\mathcal{X}_m) \in \big\{x : \phi(x) > x\big\}$, giving $\phi(\overline{x}) > \overline{x}$. The previous contradiction proves that K cannot be finite.

We proceed to show that K cannot be infinite either. The function U is continuous and, by (4), $U(x_n^C) > U(x^*)$ for all $x_n^C \in \mathcal{X}^C \cap (x^*, \overline{x}]$. Furthermore, as already pointed out above, $\mathcal{X}^C \cap (x^*, \overline{x}]$ is a compact set. Therefore,

$$\Delta := \min_{x_n^C \in \, \mathcal{X}^C \cap (x^*, \overline{x}]} U(x_n^C) - U(x^*) > 0. \tag{5}$$

Next, U being continuous and $\mathcal{X}$ compact, the function U is uniformly continuous on $\mathcal{X}$. We can thus find $\eta > 0$ such that $|U(x') - U(x)| < \Delta$ whenever $|x' - x| < \eta$. By (5), we thus have

$$U(x) > U(x^*), \text{ for all } x \text{ such that } |x - x_n^C| < \eta, \, x_n^C \in \, \mathcal{X}^C \cap (x^*, \overline{x}]. \tag{6}$$

Now, since (K, β) implements x^*, we must have $U\big(\beta(\mathcal{X}_i)\big) \leq U(x^*)$ for all $\mathcal{X}_i \in K$. So (6) shows that each member of the sequence $\mathcal{X}_1, \mathcal{X}_2, \ldots$ defined in the first part of the proof must have a length η or more. This in turn implies that said sequence can have no more than $\frac{\overline{x} - x^*}{\eta}$ terms. Yet we showed previously that this sequence cannot be finite. This contradiction completes the proof of the theorem. ■

Proof of Theorem 2: The *if* part of the theorem was proven in the text; we prove here the converse. Pick an arbitrary action x^* of the leader. Suppose that $\mathcal{Q}_{\leq}(x^*) \cap \big\{x : \phi(x) \leq x\big\} = \varnothing$. Applying Lemma 1 shows that any admissible pair (K, β) with $K \in \mathcal{K}^I$ must be such that $\beta(\mathcal{X}_i) \in \big\{x : \phi(x) \leq x\big\}$ for every $\mathcal{X}_i \in K$ containing $\overline{x}$. This, in turn, implies that every $\mathcal{K}^I$-plausible action belongs to $\mathcal{Q}_{>}(x^*)$, whence x^* cannot be $\mathcal{K}^I$-plausible. A similar argument shows that $\mathcal{Q}_{\leq}(x^*) \cap \big\{x : \phi(x) \geq x\big\} = \varnothing$ implies that x^* is not $\mathcal{K}^I$-plausible. Next, suppose that $\mathcal{Q}_{\leq}(x^*) \cap \big\{x : \phi(x) \leq x\big\}$ and $\mathcal{Q}_{\leq}(x^*) \cap \big\{x : \phi(x) \geq x\big\}$ are non-empty. Both ϕ and U

being continuous, the min and max of (3) are in this case well defined (since $\mathcal{X}$ is a compact set). Suppose that $\max \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\} < \min \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \leq x\}$, and pick

$$x^\dagger \in \Big(\max \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\}, \min \mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \leq x\}\Big). \tag{7}$$

Applying Lemma 1 shows that any admissible pair (K, β) with $K \in \mathcal{K}^I$ must be such that, for every $\mathcal{X}_i \in K$ containing $x^\dagger$, either (i) $\beta(\mathcal{X}_i) \in \{x \geq x^\dagger : \phi(x) \geq x\}$ or (ii) $\beta(\mathcal{X}_i) \in \{x \leq x^\dagger : \phi(x) \leq x\}$. So (7) gives $\beta(\mathcal{X}_i) \in \mathcal{Q}_{>}(x^*)$. It ensues that x^* cannot be $\mathcal{K}^I$-plausible. ■

Lemma A.1. *Suppose (RC1) holds. Let $K \in \mathcal{K}^I$ and $\beta : K \to \mathcal{X}$; then (K, β) constitutes an admissible pair if and only if, for all $\mathcal{X}_i \in K$:*

$$\arg\max_{x \in \mathcal{X}_i} |x - x^C| = \{\beta(\mathcal{X}_i)\}. \tag{8}$$

Proof of Lemma A.1: Let $K \in \mathcal{K}^I$ and $\beta : K \to \mathcal{X}$. We show below that if (K, β) is admissible then (8) has to hold for all $\mathcal{X}_i \in K$; the proof of the converse is analogous.

As ϕ is continuous, notice that

$$\begin{cases} \phi(x) > x & \text{for } x < x^C, \\ \phi(x) < x & \text{for } x > x^C. \end{cases} \tag{9}$$

The function $u(\cdot, R_F(x))$ being strictly concave and maximized at $\phi(x)$, it ensues that:

(a) $x < x^C$ implies $\eta(x + \epsilon, x) > 0$ for all sufficiently small $\varepsilon > 0$;

(b) $x > x^C$ implies $\eta(x - \varepsilon, x) > 0$ for all sufficiently small $\varepsilon > 0$.

If $\beta(\mathcal{X}_i) = x^C$ then (8) clearly holds. So suppose $\beta(\mathcal{X}_i) < x^C$ (the other case is analogous). Then remark (a) above gives $\eta(\beta(\mathcal{X}_i) + \epsilon, \beta(\mathcal{X}_i)) > 0$ for all sufficiently small $\varepsilon > 0$. As (K, β) is admissible, an $\varepsilon > 0$ must exist such that $(\beta(\mathcal{X}_i), \beta(\mathcal{X}_i) + \varepsilon] \subset \mathcal{X} \setminus \mathcal{X}_i$. Since $\mathcal{X}_i$ is an interval, we obtain $x \leq \beta(\mathcal{X}_i)$ for all $x \in \mathcal{X}_i$. Expression (8) thus holds, since $\beta(\mathcal{X}_i) < x^C$. ■

Alternative Proof of Proposition 2: Let the unique Cournot action be denoted by x^C. Pick an arbitrary $\mathcal{K}^I$-plausible action x^*, and let (K, β) implement x^*. By Lemma A.1, $\beta(\mathcal{X}_i) = x^C$ for all $\mathcal{X}_i \in K$ that include x^C. It follows that $U(x^*) \geq U(x^C)$. This shows that the set of $\mathcal{K}^I$-plausible actions is contained in $\mathcal{Q}_{\geq}(x^C)$.

Next, pick $x^* \in \mathcal{Q}_{\geq}(x^C) \setminus \{x^C\}$ (note that x^C is trivially $\mathcal{K}^I$-plausible). Suppose $x^* > x^C$ (the other case is analogous). Consider $K = \{\mathcal{X}_1, \mathcal{X}_2\}$, with $\mathcal{X}_1 = (\underline{x}, x^*)$ and $\mathcal{X}_2 = [x^*, \overline{x})$. Let $\beta : K \to \mathcal{X}$ be given by $\beta(\mathcal{X}_1) = x^C$, and $\beta(\mathcal{X}_2) = x^*$. As $x^* > x^C$, Lemma A.1 shows that (K, β) constitutes an admissible pair. That (K, β) implements x^* follows from Remark 1. This shows that x^* is $\mathcal{P}^I$-plausible (since $K \in \mathcal{P}^I$). So $\mathcal{Q}_{\geq}(x^C)$ is contained in the set of $\mathcal{P}^I$-plausible actions. The latter observation concludes the proof of the theorem, since any $\mathcal{P}^I$-plausible action is $\mathcal{K}^I$-plausible. ■

B Appendix of Section 4

Lemma B.1. *Suppose (RC1)–(RC3) hold. If $u_2 u_{12} > 0$, then U is increasing over $[\underline{x}, x^C]$. If $u_2 u_{12} < 0$, then U is decreasing over $[x^C, \overline{x}]$.*

Proof: We show the proof for the case in which $u_2 > 0$ and $u_{12} > 0$; the other cases are similar. Pick an arbitrary $x < x^C$, and $\varepsilon > 0$ sufficiently small that $u\big(x + \varepsilon, R_F(x)\big) > u\big(x, R_F(x)\big)$ (such an ε exists, by remark (a) in the proof of Lemma A.1). Then, R_F being non-decreasing (since $v_{12} > 0$) and $u_2 > 0$:

$$U(x + \varepsilon) = u\big(x + \varepsilon, R_F(x + \varepsilon)\big) \geq u\big(x + \varepsilon, R_F(x)\big) > u\big(x, R_F(x)\big) = U(x).$$

■

Lemma B.2. *Suppose (RC1)–(RC3) hold. Then*

$$\mathcal{S} = \big\{x : \ \eta(x^C, x) \leq 0\big\} \cap \big\{x : \ u\big(x^C, R_F(x)\big) \leq U(x^C)\big\}. \tag{10}$$

Proof: We show the proof of the lemma for the case $u_2 > 0$ and $u_{12} > 0$ (the other cases are similar). Recall that in this case $\mathcal{S} := \big\{x : \ x \leq \gamma(x) \leq x^C\big\}$.

The function R_F being in this case non-decreasing (and, indeed, increasing in a neighborhood of x^C since $y^C \in \operatorname{int}(\mathcal{Y})$) and $u_2 > 0$, notice that

$$u\big(x^C, R_F(x)\big) > u\big(x^C, R_F(x^C)\big) = U(x^C), \quad \text{for all } x > x^C.$$

So $u\big(x^C, R_F(x)\big) \leq U(x^C)$ implies $x \leq x^C$. Now consider $x \leq x^C$ such that $\eta(x^C, x) \leq 0$. We will show that $x \in \mathcal{S}$. If $x = x^C$ the previous claim is immediate, so pick $x < x^C$. The function $\eta(\cdot, x)$ is strictly concave and, by (9), maximized at $\phi(x) > x$. As $\eta(x, x) = 0 \geq \eta(x^C, x)$, we see by definition of $\gamma(x)$ that $x < \gamma(x) \leq x^C$. The right-hand side of (10) is thus contained in the set $\mathcal{S}$. The proof of the reverse inclusion is analogous. ■

Lemma B.3. *Suppose (RC1)–(RC3) hold, and $\mathcal{S} = \{x^C\}$. Then the set of $\mathcal{K}$-plausible actions coincides with the upper contour set of x^C with respect to U.*

Proof: Reason by contradiction, and suppose that some action $x^* \in \mathcal{Q}_<(x^C)$ is $\mathcal{K}$-plausible. Let (K, β) implement x^*. Choose an element $\mathcal{X}_i$ of the cover K such that $x^C \in \mathcal{X}_i$. Using Remark 1 yields $\beta(\mathcal{X}_i) \in \left\{x : \eta(x^C, x) \leq 0\right\} \cap \mathcal{Q}_{\leq}(x^*)$, and, since $x^* \in \mathcal{Q}_<(x^C)$,

$$\beta(\mathcal{X}_i) \in \left\{x : \eta(x^C, x) \leq 0\right\} \cap \mathcal{Q}_<(x^C). \tag{11}$$

In turn, (11) yields

$$u\Big(x^C, R_F\big(\beta(\mathcal{X}_i)\big)\Big) \leq u\Big(\beta(\mathcal{X}_i), R_F\big(\beta(\mathcal{X}_i)\big)\Big) = U\big(\beta(\mathcal{X}_i)\big) < U(x^C). \tag{12}$$

Coupling (11) and (12) gives

$$\beta(\mathcal{X}_i) \in \big\{x : \eta(x^C, x) \leq 0\big\} \cap \big\{x : u\big(x^C, R_F(x)\big) < U(x^C)\big\}.$$

Applying Lemma B.2, we obtain $\beta(\mathcal{X}_i) \in \mathcal{S} \setminus \{x^C\}$, contradicting $\mathcal{S} = \{x^C\}$. ■

Lemma B.4. *Suppose (RC1)–(RC3) hold. Assume $u_{12} > 0$ and $u_2 > 0$. Consider an admissible pair (K, β) which implements some action x^*. Then, if $x \in \mathcal{Q}_>(x^*)$, we have $\beta(\mathcal{X}_i) < x$ for every $\mathcal{X}_i \in K$ which contains x.*

Proof: Let $x \in \mathcal{Q}_>(x^*)$, and pick an arbitrary $\mathcal{X}_i \in K$ containing x. Reason by contradiction, and suppose that $\beta(\mathcal{X}_i) \geq x$. Then, R_F being non-decreasing (since $v_{12} > 0$) and $u_2 > 0$, we obtain

$$u\Big(x, R_F\big(\beta(\mathcal{X}_i)\big)\Big) \geq u\big(x, R_F(x)\big) > u\big(x^*, R_F(x^*)\big). \tag{13}$$

Since (K, β) is admissible, we also have

$$u\Big(\beta(\mathcal{X}_i), R_F\big(\beta(\mathcal{X}_i)\big)\Big) \geq u\Big(x, R_F\big(\beta(\mathcal{X}_i)\big)\Big). \tag{14}$$

Coupling (13) and (14) yields

$$u\Big(\beta(\mathcal{X}_i), R_F\big(\beta(\mathcal{X}_i)\big)\Big) > u\big(x^*, R_F(x^*)\big).$$

By Remark 1, the previous inequality contradicts the assumption that (K, β) implements x^*. ■

Lemma B.5. *Suppose (RC1) holds. Let (K, β) be an admissible pair. If $\beta(\mathcal{X}_i) < \min\{x^C, x\}$ for some $\mathcal{X}_i \in K$ which contains x, then $\gamma\big(\beta(\mathcal{X}_i)\big) \in \big(\beta(\mathcal{X}_i), x\big]$.*

Proof: Pick $x \in \mathcal{X}$, and $\mathcal{X}_i \in K$ containing x. Since (K, β) is admissible:

$$\eta\big(x, \beta(\mathcal{X}_i)\big) \leq 0. \tag{15}$$

Now suppose that $\beta(\mathcal{X}_i) < \min\{x^C, x\}$. In this case, the strictly concave function $\eta\big(\cdot, \beta(\mathcal{X}_i)\big)$ attains (by virtue of (9)) a maximum at $\phi\big(\beta(\mathcal{X}_i)\big) > \beta(\mathcal{X}_i)$. From (15) and the fact that $\beta(\mathcal{X}_i) < x$ we obtain (by definition of γ) $\beta(\mathcal{X}_i) < \gamma\big(\beta(\mathcal{X}_i)\big) \leq x$. ■

Proof of Theorem 3: Start with the case $\mathcal{S} = \{x^C\}$. Combining Lemma B.3 with Theorem 2 shows that in this case the sets of $\mathcal{P}^I$-, $\mathcal{K}^I$-, and $\mathcal{K}$-plausible actions are all the same. As $\mathcal{P}^I \subset \mathcal{P} \subset \mathcal{K}$, said set also coincides with the set of $\mathcal{P}$-plausible actions.

The remainder of the proof deals with the case $\mathcal{S} \supsetneq \{x^C\}$. Below, assume $u_{12} > 0$ and $u_2 > 0$ (the other cases are analogous). Recall that in this case $\mathcal{S} := \big\{x : \ x \leq \gamma(x) \leq x^C\big\}$. The function γ being continuous, $\mathcal{S}$ is a compact set. By Lemma B.1, we can thus find $\hat{x} \in \mathcal{S}$ with $\hat{x} < x^C$ and

$$U\big(\gamma(\hat{x})\big) = \min_{x \in \mathcal{S}} U\big(\gamma(x)\big). \tag{16}$$

To shorten notation, let $\hat{\gamma} := \gamma(\hat{x})$; as $\hat{x} < x^C$, note that, by definition of γ,

$$\hat{x} < \hat{\gamma} \leq x^C. \tag{17}$$

We proceed to show that (a) all actions in $\mathcal{Q}_{\geq}(\hat{\gamma})$ are $\mathcal{P}$-plausible, and (b) any $\mathcal{K}$-plausible action belongs to $\mathcal{Q}_{\geq}(\hat{\gamma})$.

All actions in $\mathcal{Q}_{\geq}(\hat{\gamma})$ are $\mathcal{P}$-plausible. We know by Theorem 2 that all actions in $\mathcal{Q}_{\geq}(x^C)$ are $\mathcal{P}^I$-plausible. So pick an action $x^* \in \mathcal{Q}_{\geq}(\hat{\gamma}) \setminus \mathcal{Q}_{\geq}(x^C)$ (if there exists none, we are done). Define

$$\mathcal{X}_1 := \{\hat{x}\} \cup \mathcal{Q}_{>}(x^*),$$

and let P denote the partition of $\mathcal{X}$ made up of $\mathcal{X}_1$, and only singletons besides $\mathcal{X}_1$. Lastly, let $\beta : P \to \mathcal{X}$ be given by $\beta(\mathcal{X}_1) = \hat{x}$ and $\beta(\{x\}) = x$ for all $x \in \mathcal{X} \setminus \mathcal{X}_1$. We now show that

(P, β) constitutes an admissible pair; notice that this amounts to showing that

$$\eta(\tilde{x}, \hat{x}) \leq 0, \quad \text{for all } \tilde{x} \in \mathcal{X}_1. \tag{18}$$

As $x^* \in \mathcal{Q}_{\geq}(\hat{\gamma})$, any $\tilde{x} \in \mathcal{Q}_{>}(x^*)$ belongs to $\mathcal{Q}_{\geq}(\hat{\gamma})$. On the other hand, since $\hat{\gamma} \leq x^C$ (see (17)), Lemma B.1 shows that every $\tilde{x} \in \mathcal{Q}_{>}(x^*)$ satisfies $\tilde{x} \geq \hat{\gamma}$. Now, the function $\eta(\cdot, \hat{x})$ is strictly concave, with $\eta(\hat{x}, \hat{x}) = \eta(\hat{\gamma}, \hat{x}) = 0$; it thus follows from (17) that $\eta(\tilde{x}, \hat{x}) \leq 0$ for all $\tilde{x} \geq \hat{\gamma}$. Combining the previous observations establishes (18); so (P, β) is admissible.

Finally, coupling (17) and Lemma B.1 yields $U(\hat{\gamma}) > U(\hat{x})$, giving in turn $U(x^*) > U(\hat{x}) = U\big(\beta(\mathcal{X}_1)\big)$ (since $x^* \in \mathcal{Q}_{\geq}(\hat{\gamma})$). Using Remark 1 now shows that (P, β) implements x^*, since $\mathcal{X} \setminus \mathcal{X}_1 \subset \mathcal{Q}_{\leq}(x^*)$.

All $\mathcal{K}$-plausible actions belong to $\mathcal{Q}_{\geq}(\hat{\gamma})$. Reason by contradiction, and suppose that some $\mathcal{K}$-plausible action x^* belongs to $\mathcal{Q}_{<}(\hat{\gamma})$. Combining (17), Lemma B.1, and the fact that U is continuous shows that we can find an action, say $x^\dagger$, such that:

$$x^\dagger < \hat{\gamma}, \tag{19}$$

and

$$x^\dagger \in \mathcal{Q}_{>}(x^*) \cap \mathcal{Q}_{<}(\hat{\gamma}). \tag{20}$$

Now consider a pair (K, β) which implements x^*, and $\mathcal{X}_i$ an element of the cover K containing $x^\dagger$. By virtue of (20), applying Lemma B.4 shows that

$$\beta(\mathcal{X}_i) < x^\dagger. \tag{21}$$

On the other hand, (17) and (19) show that

$$x^\dagger < \hat{\gamma} \leq x^C.$$

Hence, Lemma B.5 gives

$$\beta(\mathcal{X}_i) < \gamma\big(\beta(\mathcal{X}_i)\big) \leq x^\dagger < \hat{\gamma} \leq x^C. \tag{22}$$

We thus obtain, firstly,

$$\beta(\mathcal{X}_i) \in \mathcal{S}, \tag{23}$$

and, secondly (using Lemma B.1),

$$U\Big(\gamma\big(\beta(\mathcal{X}_i)\big)\Big) < U(\hat{\gamma}). \tag{24}$$

The combination of (23) and (24) contradicts (16). Therefore, every $\mathcal{K}$-plausible action must belong to $\mathcal{Q}_{\geq}(\hat{\gamma})$. ■

Proof of Proposition 3: By definition of γ: $\eta\big(\gamma(x), x\big) = 0$ for all x in some neighborhood O of x^C. We thus have

$$u\big(\gamma(x), R_F(x)\big) = u\big(x, R_F(x)\big), \quad \forall x \in O.$$

Differentiating the previous expression with respect to x yields

$$u_1\big(\gamma(x), R_F(x)\big)\gamma'(x) + u_2\big(\gamma(x), R_F(x)\big)R_F'(x) = u_1\big(x, R_F(x)\big) + u_2\big(x, R_F(x)\big)R_F'(x),$$

and, therefore,

$$\gamma'(x) = \frac{u_1\big(x, R_F(x)\big) + R_F'(x)\big[u_2\big(x, R_F(x)\big) - u_2\big(\gamma(x), R_F(x)\big)\big]}{u_1\big(\gamma(x), R_F(x)\big)}, \quad \forall x \in O \setminus \{x^C\}. \tag{25}$$

The numerator and denominator on the right-hand side of (25) tend to 0 as $x \to x^C$. Then, by virtue of L'Hospital's rule and using the fact that $\gamma(x) \to x^C$ as $x \to x^C$:

$$\lim_{x \to x^C} \gamma'(x) = \lim_{x \to x^C} \frac{u_{11}\big(x, R_F(x)\big) + 2u_{12}\big(x, R_F(x)\big)R_F'(x) - u_{12}\big(x, R_F(x)\big)R_F'(x)\gamma'(x)}{u_{11}\big(\gamma(x), R_F(x)\big)\gamma'(x) + u_{12}\big(\gamma(x), R_F(x)\big)R_F'(x)}. \tag{26}$$

On the other hand, in a neighborhood of $y = y^C$:

$$R_L'(y) = \frac{-u_{12}\big(R_L(y), y\big)}{u_{11}\big(R_L(y), y\big)}.$$

Therefore,

$$R_L'(y^C) = \frac{-u_{12}(x^C, y^C)}{u_{11}(x^C, y^C)} = \lim_{x \to x^C} \frac{-u_{12}\big(x, R_F(x)\big)}{u_{11}\big(x, R_F(x)\big)} = \lim_{x \to x^C} \frac{-u_{12}\big(\gamma(x), R_F(x)\big)}{u_{11}\big(\gamma(x), R_F(x)\big)}. \tag{27}$$

Combining (27) with (26) gives

$$\gamma'(x^C) = \frac{1 - 2R'_L(y^C)R'_F(x^C) + R'_L(y^C)R'_F(x^C)\gamma'(x^C)}{\gamma'(x^C) - R'_L(y^C)R'_F(x^C)}.$$

So $\gamma'(x^C)$ is a solution of

$$Z(Z - 2\alpha) = 1 - 2\alpha,$$

where $\alpha := R'_L(y^C)R'_F(x^C)$. So either $\gamma'(x^C) = 1$ or $\gamma'(x^C) = 2\alpha - 1$, whence $\gamma'(x^C) > 0$ if $R'_L(y^C)R'_F(x^C) > 1/2$.

Now suppose that $u_{12}u_2 > 0$ (the other case is similar), so that $\mathcal{S} = \{x: \ x \leq \gamma(x) \leq x^C\}$. If $R'_L(y^C)R'_F(x^C) > 1/2$, then $\gamma'(x^C) > 0$. This in turn implies the existence of $x < x^C$ such that $x < \gamma(x) < x^C$. Such an x belongs to $\mathcal{S}$, so Lemma B.1 enables us to conclude that $\underline{U} < U(x^C)$. ■

Proof of Proposition 4: Just notice that the partition P in the part of the proof of Theorem 3 showing that all actions in $\mathcal{Q}_{\geq}(\hat{\gamma})$ are $\mathcal{P}$-plausible satisfies $P \in \mathcal{P}^{I+}$. ■

C Appendix of Section 5

All the results in this appendix refer to the duopoly example of Section 2.3. Subsection C.1 characterizes the sets of plausible quantities. Subsection C.2 proves Proposition 6.

We refer here to the set of $\mathcal{P}^I$-plausible quantities as $\mathcal{X}^{\mathcal{P}^I}$. We use similar notation for the sets of $\mathcal{K}^I$-plausible, $\mathcal{P}$-plausible and $\mathcal{K}$-plausible quantities. Whenever a set $\mathcal{X}^z$ has a minimum (respectively, a maximum) we denote it $\underline{x}^z$, (resp. $\overline{x}^z$). For instance, $\underline{x}^{\mathcal{P}^I}$ denotes the smallest $\mathcal{P}^I$-plausible quantity. We define the following functions:

$$r^*(d) := 2-\sqrt{2}(1-d);$$
$$r^{**}(d) := 2-\left(\sqrt[3]{\frac{\sqrt{57}}{9}+1}\right)(1-d)-\frac{2(1-d)}{3\sqrt[3]{\frac{\sqrt{57}}{9}+1}};$$
$$r^{***}(d) := \frac{1}{2}\left(3-\sqrt{5}+(1+\sqrt{5})d\right);$$
$$r^{\dagger}(d) := 2-\left(\frac{\sqrt[3]{3(9-\sqrt{78})}}{3}+\frac{1}{\sqrt[3]{3(9-\sqrt{78})}}\right)(1-d);$$
$$r^{\dagger\dagger}(d) := 2-\sqrt{3}(1-d);$$
$$r^{\dagger\dagger\dagger}(d) := 2+\left(\frac{1-\sqrt[3]{80-9\sqrt{79}}}{3}-\frac{1}{3\sqrt[3]{80-9\sqrt{79}}}\right)(1-d).$$

A firm acting as a monopolist would choose quantity $x^M := \frac{1}{2-r}$.

C.1 Plausible Quantities

The unique best response of the follower to x, and the leader payoff from x when the follower best responds to x are given, respectively, by

$$R_F(x) = \begin{cases} \frac{1-(1-d)x}{2-r} & \text{if } x \le \frac{1}{1-d}, \\ 0 & \text{if } x > \frac{1}{1-d}, \end{cases} \quad \text{and } U(x) = \begin{cases} \frac{2(1-r+d)x-((2-r)^2-2(1-d)^2)x^2}{2(2-r)} & \text{if } x \le \frac{1}{1-r}, \\ x-\left(1-\frac{r}{2}\right)x^2 & \text{if } x > \frac{1}{1-d}. \end{cases}$$

Function ϕ takes the form:

$$\phi(x) = \begin{cases} 0 & \text{if } x \le \frac{r-(d+1)}{(1-d)^2}, \\ \frac{d+1-r+(1-d)^2x}{(2-r)^2} & \text{if } \frac{r-(d+1)}{(1-d)^2} < x < \frac{1}{1-d}, \\ x^M & \text{if } x \ge \frac{1}{1-d}. \end{cases}$$

We characterize next the Cournot and the Stackelberg quantities.

Lemma C.1. *The set of Cournot quantities is as follows:*

$$\mathcal{X}^C = \begin{cases} \left\{\frac{1}{3-r-d}\right\} & \text{if } r < d+1, \\ \left[0, x^M\right] & \text{if } r = d+1, \\ \left\{0, \frac{1}{3-r-d}, x^M\right\} & \text{if } r > d+1. \end{cases}$$

Proof:

(i) If $r < d+1$, then

$$\frac{r-(d+1)}{(1-d)^2} < 0 \text{ and } \frac{1}{1-d} > \frac{2}{2-r},$$

hence $\mathcal{X}^C = \{x^*\}$ where $x = x^*$ solves

$$\frac{d+1-r+(1-d)^2 x}{(2-r)^2} = x. \tag{28}$$

(ii) If $r = d+1$, then $\phi(x) = x \iff x \leq \frac{1}{1-d}$, and $x^M = \frac{1}{1-d}$.

(iii) If $r > d+1$, then

$$\frac{2}{2-r} > \frac{1}{1-d} > \frac{r-(d+1)}{(1-d)^2} > 0,$$

hence set $\mathcal{X}^C$ includes only 0, x^M, and the solution to (28).

■

In this appendix, $x_1^C = 0$, $x_2^C = \frac{1}{3-r-d}$, $x_3^C = x^M$ and $x^C = x_2^C$.

Lemma C.2. *The Stackelberg quantity, denoted x^S, is as follows:*

$$x^S = \begin{cases} \frac{d+1-r}{(2-r)^2-2(1-d)^2} & \text{if } r < r^{***}(d), \\ \frac{1}{1-d} & \text{if } r^{***}(d) \leq r \leq d+1, \\ x^M & \text{if } r > d+1. \end{cases}$$

Proof: If $r \leq d+1$, then $U'(x) < 0$ for any $x > \frac{1}{1-d}$, hence $x^S = [0, \frac{1}{1-d}]$. Note that (i) U is a quadratic function over this interval, (ii) $U'(0) > 0$, and (iii) $U'\left(\frac{d+1-r}{(2-r)^2-2(1-d)^2}\right) = 0$. Thus,

$$\arg\max_{x\in\mathcal{X}} U(x) \in \left\{\frac{1}{1-d}, \frac{d+1-r}{(2-r)^2-2(1-d)^2}\right\}.$$

A few steps of algebra yield:

$$U\left(\frac{1}{1-d}\right) \geq U\left(\frac{d+1-r}{(2-r)^2-2(1-d)^2}\right) \iff r \geq r^{***}(d).$$

One can also check that: $r \in [0, r^{***}(d)] \Rightarrow \frac{d+1-r}{(2-r)^2-2(1-d)^2} \in [0, \frac{1}{1-d}]$. Thus, $x^S = \frac{d+1-r}{(2-r)^2-2(1-d)^2}$ for $r < r^{***}(d)$ and $x^S = \frac{1}{1-d}$ for $r \in [r^{***}(d), d+1]$. Finally, if $r > d+1$ then $R_F(x^M) = 0$ and therefore $\arg\max_{x\in\mathcal{X}} U(x) = x^M$. ■

Next, we characterize the sets of plausible quantities.

Lemma C.3. *The set of $\mathcal{P}^I$-plausible quantities is as follows:*

$$\mathcal{X}^{\mathcal{P}^I} = \begin{cases} \left[x^C, \frac{(2-r)^2}{(-r-d+3)((2-r)^2-2(1-d)^2)}\right] & \text{if } r < r^{**}(d), \\ \left[x^C, \frac{\sqrt{(1-d)(-2r-d+5)}-r-d+3}{(2-r)(-r-d+3)}\right] & \text{if } r^{**}(d) \leq r < d+1, \\ \mathcal{X} & \text{if } r = d+1, \\ \{x_1^C\} \cup \left[\frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}, x_2^C\right] \cup \left[x_3^C, \frac{2}{2-r}\right] & \text{if } r > d+1. \end{cases}$$

Proof: (i) If $r < d+1$, then $\mathcal{X}^C = \{x^C\}$, hence Proposition 2 ensures $\mathcal{X}^{\mathcal{P}^I} = \mathcal{Q}_{\geq}(x^C)$. For $r < d+1$, then (i) $U'(x) < 0$ for any $x \geq \frac{1}{1-d}$, and (ii) over the interval $[0, \frac{1}{1-d}]$, function U is either non decreasing or concave, or both. Function U is thus quasi-concave. As $U'(x^C) > 0$, then $\mathcal{Q}_{\geq}(x^C) = [x^C, \overline{x}^{\mathcal{P}^I}]$, where $\overline{x}^{\mathcal{P}^I}$ satisfies $\overline{x}^{\mathcal{P}^I} > x^C$ and $U(\overline{x}^{\mathcal{P}^I}) = U(x^C)$. It is easy to verify that $x^C < (1+d)^{-1}$, while $r > r^{**}(d) \iff \overline{x}^{\mathcal{P}^I} > (1+d)^{-1}$. A few steps of algebra thus yield the expressions for $\overline{x}^{\mathcal{P}^I}$.

(ii) Lemma C.1 ensures that if $r = d+1$, then $\mathcal{X}^C = [0, x^M]$. For all $x > x^M$, it is the case that $x > \phi(x) = x^M$. Theorem 1 thus ensures $\mathcal{X}^{\mathcal{P}^I} = \mathcal{X}$.

(iii) If $r > d+1$, the characterization of the set $\mathcal{X}^{\mathcal{P}^I}$ follows directly from Theorem 1 and properties of ϕ. Note in particular that

- if $x^* \in \{x_1^C\} \cup \left[\frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}, x_2^C\right] \cup \left[x_3^C, \frac{2}{2-r}\right]$, then $(\phi(x^*)-x^*)(x_1^C-x^*) \geq 0$ hence $x^* \in \mathcal{X}^{\mathcal{P}^I}$;
- if instead $x^* \notin \{x_1^C\} \cup \left[\frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}, x_2^C\right] \cup \left[x_3^C, \frac{2}{2-r}\right]$, then $(\phi(x^*)-x^*)(x_i^C-x^*) < 0$ for $i = 1$, 2 and 3, hence $x^* \notin \mathcal{X}^{\mathcal{P}^I}$.

■

Lemma C.4. *The set of $\mathcal{K}^I$-plausible quantities is as follows:*

$$\mathcal{X}^{\mathcal{K}^I} = \begin{cases} \mathcal{X}^{\mathcal{P}^I} & \text{if } r \leq d+1, \\ \{0\} \cup \left[\frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}, \frac{2}{2-r}\right] & \text{if } r > d+1. \end{cases}$$

Proof: (i) If $r < d+1$, conditions (RC1)–(RC3) hold, hence $\mathcal{X}^{\mathcal{K}^I} = \mathcal{X}^{\mathcal{P}^I}$ by Proposition 2.

(ii) If $r = d+1$, then $\mathcal{X}^{\mathcal{P}^I} = \mathcal{X}$ (Lemma C.3). As $\mathcal{X} \supseteq \mathcal{X}^{\mathcal{K}^I}$ and $\mathcal{X}^{\mathcal{K}^I} \supseteq \mathcal{X}^{\mathcal{P}^I}$, then $\mathcal{X}^{\mathcal{K}^I} = \mathcal{X}^{\mathcal{P}^I}$.

(iii) Suppose $r > d+1$. If $x^* \in \left(0, \frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}\right)$, then $\mathcal{Q}_{\leq}(x^*) \cap \{x : \phi(x) \geq x\} = \varnothing$; Theorem 2 ensures $x^* \notin \mathcal{X}^{\mathcal{K}^I}$. If instead $x^* \in \{0\} \cup \left[\frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}, \frac{2}{2-r}\right]$, then $x^* \in \mathcal{Q}_{\geq}(x_1^C)$; Corollary 2 ensures $x^* \in \mathcal{X}^{\mathcal{K}^I}$.

■

Lemma C.5. *The set of $\mathcal{K}$-plausible quantities is as follows:*

$$\mathcal{X}^{\mathcal{K}} = \begin{cases} \left[\frac{2(d+1-r)}{(2-r)^2}, \frac{(2-r)^2+\sqrt{(2-r)^4-8(1-d)^2(d+1-r)^2}}{(2-r)^3}\right] & \text{if } r^*(d) \leq r < d+1, \\ \mathcal{X}^{\mathcal{K}^I} & \text{otherwise.} \end{cases}$$

Proof: (i) If $r < d+1$, conditions (RC1)–(RC3) hold, and Theorem 3 applies. In particular, if $r < r^*(d)$, then $\mathcal{S} = \{x^C\}$, and therefore $\mathcal{X}^{\mathcal{K}} = \mathcal{X}^{\mathcal{P}^I}$, which in turn implies $\mathcal{X}^{\mathcal{K}} = \mathcal{X}^{\mathcal{K}^I}$. If instead $r \geq r^*(d)$, then $\mathcal{S} = [0, x^C]$. Note that $x^C < \frac{1}{1-d}$, hence

$$\gamma(x^*) = \frac{2(1+d-r) - x(2-r)^2 + 2x(1-d)^2}{(2-r)^2}, \text{ for all } x^* \in [0, x^C].$$

One can then verify that $0 = \arg\min_{x \in [0, x^C]} U(\gamma(x))$, and $\gamma(0) = \frac{2(d+1-r)}{(2-r)^2}$. Solving the equation $U(x) = U(\gamma(0))$, and noting that U is quasi-concave, yields

$$\mathcal{X}^{\mathcal{K}} = \left[\gamma(0), \frac{(2-r)^2 + \sqrt{(2-r)^4 - 8(1-d)^2(d+1-r)^2}}{(2-r)^3}\right].$$

(ii) If $r = d+1$, then $\mathcal{X}^{\mathcal{K}^I} = \mathcal{X}$ (see Lemma C.4). As $\mathcal{X}^{\mathcal{K}} \supseteq \mathcal{X}^{\mathcal{K}^I}$ and $\mathcal{X} \supseteq \mathcal{X}^{\mathcal{K}}$, we conclude that $\mathcal{X}^{\mathcal{K}^I} = \mathcal{X}^{\mathcal{K}}$.

(iii) If $r > d+1$, Lemma C.4 ensures that $\mathcal{Q}_{\geq}(0) = \mathcal{X}^{\mathcal{K}^I}$. As $u(0,y) = 0$ for any $y \in \mathcal{X}$, clearly $\mathcal{Q}_{<}(0) \notin \mathcal{X}^{\mathcal{K}}$; thus, $\mathcal{X}^{\mathcal{K}} = \mathcal{X}^{\mathcal{K}^I}$.

■

The following remark is easy to verify.

Remark C.1. *If $r > d+1$, then $\mathcal{Q}_{\geq}(0) = \{0\} \cup \left[\frac{2(r-d-1)}{2(1-d)^2-(2-r)^2}, \frac{2}{2-r}\right]$. If instead $r \leq d+1$, then $\mathcal{Q}_{\geq}(0) = \mathcal{X}$.*

Lemma C.6. *The sets of $\mathcal{P}$-plausible, $\mathcal{P}^{I+}$-plausible and $\mathcal{K}$-plausible quantities coincide:*

$$\mathcal{X}^{\mathcal{P}} = \mathcal{X}^{\mathcal{P}^{I+}} = \mathcal{X}^{\mathcal{K}}.$$

Proof: We focus first on the set $\mathcal{X}^{\mathcal{P}^{I+}}$. Recall that $\mathcal{X}^{\mathcal{P}^{I+}} \subseteq \mathcal{X}^{\mathcal{K}}$.

(i) Consider the case $r \geq d+1$. Lemmata C.3, C.4 and C.5 together with Remark C.1 imply that $\mathcal{X}^{\mathcal{K}} = \mathcal{Q}_{\geq}(0)$.

Take any action $x^* \in \mathcal{Q}_{\geq}(0)$. To see that $x^* \in \mathcal{X}^{\mathcal{P}^{I+}}$, let $\mathcal{X}_1 = \{x^*\}$, $\mathcal{X}_2 = \mathcal{X} \setminus \{x^*\}$, $\beta(\mathcal{X}_1) = x^*$, $K = \{\mathcal{X}_1, \mathcal{X}_2\}$ and define $\beta : K \to \mathcal{X}$ as follows: $\beta(\mathcal{X}_1) = x^*$ and $\beta(\mathcal{X}_2) = 0$. Then $K \in \mathcal{X}^{\mathcal{P}^{I+}}$, and the pair (K, β) implements x^*. Therefore $x^* \in \mathcal{X}^{\mathcal{P}^{I+}}$, which implies that $\mathcal{X}^{\mathcal{P}^{I+}} = \mathcal{X}^{\mathcal{K}}$.

(ii) If $r < d+1$, then Proposition 4 ensures $\mathcal{X}^{\mathcal{P}^{I+}} = \mathcal{X}^{\mathcal{K}}$.

Finally, as $\mathcal{X}^{\mathcal{K}} = \mathcal{X}^{\mathcal{P}^{I+}} \subseteq \mathcal{X}^{\mathcal{P}} \subseteq \mathcal{X}^{\mathcal{K}}$, then $\mathcal{X}^{\mathcal{P}} = \mathcal{X}^{\mathcal{K}}$. ■

We conclude with an immediate corollary of Lemma C.5 that will prove useful in the next subsection.

Corollary C.1. *The smallest and the largest $\mathcal{K}$-plausible actions correspond to:*

$$\left\{\underline{x}^{\mathcal{K}}, \overline{x}^{\mathcal{K}}\right\} = \begin{cases} \left\{\underline{x}^{\mathcal{P}^I}, \overline{x}^{\mathcal{P}^I}\right\} = \left\{0, \frac{2}{2-r}\right\} & \text{if } r \geq d+1, \\ \left\{\underline{x}^{\mathcal{P}}, \overline{x}^{\mathcal{P}}\right\} = \left\{\frac{2(d+1-r)}{(2-r)^2}, \frac{(2-r)^2+\sqrt{(2-r)^4-8(1-d)^2(d+1-r)^2}}{(2-r)^3}\right\} & \text{if } r^*(d) < r < d+1, \\ \left\{\underline{x}^{\mathcal{P}^I}, \overline{x}^{\mathcal{P}^I}\right\} = \left\{x^C, \frac{\sqrt{(1-d)(-2r-d+5)}-r-d+3}{(2-r)(-r-d+3)}\right\} & \text{if } r^{**}(d) < r < r^*(d), \\ \left\{\underline{x}^{\mathcal{P}^I}, \overline{x}^{\mathcal{P}^I}\right\} = \left\{x^C, \frac{(2-r)^2}{(-r-d+3)((2-r)^2-2(1-d)^2)}\right\} & \text{if } r \leq r^{**}(d). \end{cases}$$

C.2 The Designer Problem

We prove each of the three parts of Proposition 6 separately. To prove the first part, we need the next two lemmata, where we characterize the solution the following problems

$$\max x + y \quad \text{s.t. } (x, y) \text{ is } \mathcal{K}\text{-plausible}, \tag{29}$$

and

$$\min xy \quad \text{s.t. } (x, y) \text{ is } \mathcal{K}\text{-plausible}. \tag{30}$$

Lemma C.7. *The unique solution of* (29) *is* $(\overline{x}^{\mathcal{K}}, R_F(\overline{x}^{\mathcal{K}}))$.

Proof: Outcome (x, y) is $\mathcal{K}$-plausible only if $y = R_F(x)$, and

$$x + R_F(x) = \begin{cases} \frac{1+(1+d-r)x}{2-r}, & \text{if } x < \frac{1}{1-d}, \\ x, & \text{if } x \geq \frac{1}{1-d}. \end{cases}$$

If $r \leq d+1$, then $x+R_F(x)$ is non-decreasing in x, and therefore $\overline{x}^{\mathcal{K}} \in \arg\max_{x\in\mathcal{X}^{\mathcal{K}}} x+R_F(x)$. If $r > d + 1$, then: (i) $x + R_F(x)$ is quasi-convex in x, (ii) $\overline{x}^{\mathcal{K}} = \frac{2}{2-r}$ (Corollary C.1), and (iii) $0 + R_F(0) = \frac{1}{2-r} < \overline{x}^{\mathcal{K}} \leq \overline{x}^{\mathcal{K}} + R_F(\overline{x}^{\mathcal{K}})$. The lemma follows. ■

Lemma C.8. *If* $r \geq 2d$*, the unique solution of* (30) *is* $(\overline{x}^{\mathcal{K}}, R_F(\overline{x}^{\mathcal{K}}))$.

Proof: If $r \geq d + 1$ then $\overline{x}^{\mathcal{K}} = \frac{2}{2-r}$. Note that $r \geq 2d \iff > \frac{2}{2-r} \geq \frac{1}{1-d} \iff R_F(\frac{2}{2-r}) = 0$. The proof of Lemma C.3 shows that $\overline{x}^{\mathcal{P}^I} \geq \frac{1}{1-d}$ if $r \in (r^{**}(d), d + 1)$. As $\overline{x}^{\mathcal{K}} \geq \overline{x}^{\mathcal{P}^I}$, then $r \in (r^{**}(d), d + 1) \Rightarrow R_F(\overline{x}^{\mathcal{K}}) = 0$. Let $f(x) := xR_F(x)$. As $f(x) \geq 0$ for all $x \in \mathcal{X}$, we conclude that $\overline{x}^{\mathcal{K}} = \arg\min_{x\in\mathcal{X}^{\mathcal{K}}} f(x)$ for $r > r^{**}(d)$. Finally, if $r \leq r^{**}(d)$, then

$$\{\underline{x}^{\mathcal{K}}, \overline{x}^{\mathcal{K}}\} = \left\{x^C, \frac{(2-r)^2}{(3-r-d)\left((2-r)^2 - 2(1-d)^2\right)}\right\}.$$

Function f is convex, and

$$f\left(\frac{(2-r)^2}{(3-r-d)\left((2-r)^2 - 2(1-d)^2\right)}\right) \geq f(x^C) \iff r \geq 2d.$$

The lemma follows. ■

Proof of Proposition 6, part (i). Any $\mathcal{K}$-plausible quantity x is associated with consumer surplus:

$$CS(x, R_F(x)) = \frac{(x + R_F(x))^2}{2} - dxR_F(x).$$

Let $g(x) := CS(x, R_F(x))$. If $r \geq 2d$, then Lemmata C.7 and C.8 together ensure that $\overline{x}^{\mathcal{K}} = \arg\max_{x \in \mathcal{X}^{\mathcal{K}}} g(x)$.

Suppose that $r < 2d$, so that $\frac{2}{2-r} < \frac{1}{1-d}$. We now prove that $g(\cdot)$ is increasing over the set $\mathcal{X}^{\mathcal{K}}$. First note that, in this parameter region, $g(x) = a_0 + a_1 x + a_2 x^2$, where

$$a_0 := \frac{1}{2(2-r)^2}, \quad a_1 := \frac{(1-r)(1-d)}{(2-r)^2}, \text{ and } \ a_2 := \frac{(d+1-r)^2 + 2(2-r)(1-d)d}{2(2-r)^2}.$$

Function g is then convex, and $\arg\min_x g(x) = \frac{-a_1}{2a_2}$. As $2d < r^{**}(d)$, then $r < 2d$ implies $\underline{x}^{\mathcal{K}} = x^C$. Note that

$$x^C > \frac{-a_1}{2a_2} \quad \iff \quad \frac{(2-r)(2-d)(d+1-r)}{(3-r-d)\left((d+1-r)^2 + 2(2-r)(1-d)d\right)} > 0.$$

This inequality holds, hence $g(\cdot)$ is increasing over the set $\mathcal{X}^{\mathcal{K}}$. ■

To prove the second part of Proposition 6 we need the following lemma.

Lemma C.9. *For any* $d \in [0, 1)$,

$$2d < r^{\dagger\dagger}(d) < r^{\dagger\dagger\dagger}(d) < r^{\dagger}(d) < r^{*}(d) < d + 1.$$

Proof: Functions $2d$, $r^{\dagger\dagger}(d)$, $r^{\dagger\dagger\dagger}(d)$, $r^{\dagger}(d)$, $r^{***}(d)$, and $1 + d$ are linear and take value 2 for $d = 1$. To prove the lemma it is therefore sufficient to verify that their slopes are ordered appropriately. The slopes are shown in Table 1 ■

Function	Slope
$2d$	2
$r^{\dagger\dagger}(d)$	$\sqrt{3} \approx 1.732$
$r^{\dagger\dagger\dagger}(d)$	$\frac{1}{3}\sqrt[3]{80 - 9\sqrt{79}} - \frac{1}{3} + \frac{1}{3\sqrt[3]{80-9\sqrt{79}}} \approx 1.538$
$r^{\dagger}(d)$	$\frac{\sqrt[3]{3(9-\sqrt{78})}}{3} + \frac{1}{\sqrt[3]{3(9-\sqrt{78})}} \approx 1.518$
$r^{*}(d)$	$\sqrt{2} \approx 1.414$
$d + 1$	1

Table 1: Slopes of functions from Lemma C.9

Proof of Proposition 6, part (ii). Any $\mathcal{K}$-plausible quantity x is associated with producer surplus

$$PS(x,R_F(x)) = (x+R_F(x)) - \left(1-\frac{r}{2}\right)(x+R_F(x))^2 - (r-2d)xR_F(x),$$
$$= \begin{cases} \frac{1-2rx+4dx-x^2+4rx^2-r^2x^2-6dx^2+3d^2x^2}{2(2-r)} & \text{if } x<\frac{1}{1-d}, \\ x-\left(1-\frac{r}{2}\right)x^2 & \text{if } x\geq\frac{1}{1-d}. \end{cases} \quad (31)$$

Let $h(x) := PS(x,R_F(x))$. If $r>d+1$, then $x_1^C\in\mathcal{X}^{\mathcal{K}}$, $x_3^C\in\mathcal{X}^{\mathcal{K}}$, $R_F(x_3^C)=x_1^C=0$ and $R_F(x_1^C)=x_3^C$. As

$$x_3^C+R_F(x_3^C)=x_1^C+R_F(x_1^C)=\arg\max_{x\in\mathcal{X}} x-\left(1-\frac{r}{2}\right)x^2,$$

and $x_3^C R_F(x_3^C)=x_1^C R_F(x_1^C)=0$, we conclude that both x_1^C and x_3^C maximize producer surplus among $\mathcal{K}$-plausible quantities. The argument can be extended to the case $r=d+1$.

Suppose now that $r<d+1$. It is easy to check that $h(\cdot)$ is decreasing over the interval $[\frac{1}{1-d},\frac{2}{2-r}]$. Note that $\frac{1}{1-d}\in\mathcal{X}^{\mathcal{K}}$. For $x\in[0,\frac{1}{1-d}]$ instead, $g(x)=a_0+a_1x+a_2x^2$, where

$$a_0 := \frac{1}{2(2-r)},\quad a_1 := -\frac{r-2d}{2-r}<0, \text{ and }\quad a_2 := \frac{-r^2+4r+3d^2-6d-1}{2(2-r)}.$$

Specifically, $a_2>0$ if and only if $r>r^{\dagger\dagger}(d)$. Therefore for $r\in[r^{\dagger\dagger}(d),1+d)$, the function g takes the highest value either at $\frac{1}{1-d}$, or at $\underline{x}^{\mathcal{K}}$. Note that $g\left(\frac{1}{1-d}\right)=PS^1:=\frac{r-2d}{2(1-d)^2}$. In order to characterize $g(\underline{x}^{\mathcal{K}})$, we distinguish two cases. If $r<r^*(d)$, then $\underline{x}^{\mathcal{K}}=x^C$. Note that $g\left(\frac{1}{1-d}\right)>g(x^C) \iff r>r^\dagger(d)$. Lemma C.9 ensures that $r^{\dagger\dagger}(d)<r^\dagger(d)$. If instead $r\geq r^*(d)$, then $\underline{x}^{\mathcal{K}}=\frac{2(d+1-r)}{(2-r)^2}$, and

$$g\left(\frac{1}{1-d}\right)>g\left(\frac{2(d+1-r)}{(2-r)^2}\right) \iff$$
$$A\cdot\left(r^3+r^2d-7r^2-4rd+16r-6d^3+18d^2-14d-6\right)>0,$$

where

$$A := \frac{(d+1-r)\,(r^2-2rd-2r+2d^2+2)}{2(2-r)^5(1-d)^2}>0.$$

This inequality holds in the interval $[r^{\dagger\dagger\dagger}(d),1+d]$. As $r^*(d)>r^{\dagger\dagger\dagger}(d)$ (Lemma C.9), we conclude that

$$\frac{1}{1-d} = \arg\max_{x\in\mathcal{X}^{\mathcal{K}}} g(x) \text{ for } r \in [r^{\dagger}(d), 1+d],$$

and

$$x^C = \arg\max_{x\in\mathcal{X}^{\mathcal{K}}} g(x) \text{ for } r \in [r^{\dagger\dagger}(d), r^{\dagger}(d)].$$

Consider next $r \in [2d, r^{\dagger\dagger}(d))$. For these parameter values the function g is concave over the interval $[0, \frac{1}{1-d}]$. The global maximum obtains at $x = -a_1/2a_2 \leq 0$. Therefore $\arg\max_{x\in\mathcal{X}^{\mathcal{K}}} g(x) = \underline{x}^{\mathcal{K}}$. As $r^{\dagger\dagger}(d) < r^*(d)$, then $\underline{x}^{\mathcal{K}} = x^C$.

Finally, consider the case $r < 2d$. For these parameter values, the function g is concave over the interval $x \in [0, (1-d)^1]$, and reaches its maximum at

$$\frac{-a_1}{2a_2} = \frac{-(r-2d)}{r^2 - 4r - 3d^2 + 6d + 1} > 0.$$

As $r < 2d$, then (i) $\underline{x}^{\mathcal{K}} = x^C$, and (ii) $x^C > \frac{-a_1}{2a_2} \iff r < r^{\dagger\dagger}(d)$. Noting that $r^{\dagger\dagger}(d) > 2d$ (Lemma C.9) concludes the proof. ■

Proof of Proposition 6, part (iii): Any $\mathcal{K}$-plausible quantity x is associated with total welfare

$$W(x, R_F(x)) = CS(x, R_F(x)) + PS(x, R_F(x)) = Q(x) - \frac{1-r}{2}Q(x)^2 - (r-d)xR_F(x),$$

where $Q(x) = x + R_F(x)$ is the total quantity.

Let us first consider the case $r \geq 2d$. Define $f(Q) := Q - \frac{1}{2}(1-r)Q^2$. Whenever $r \geq 0$, the function f is increasing over the interval $\mathcal{X}$. To see this, note that (i) if $r > 1$ then f is convex and $\arg\min f = (1-r)^{-1} < 0$; (ii) if $r = 1$, then f is increasing for all Q; (iii) if $r < 1$ then, f is concave and $\arg\max f = (1-r)^{-1} > \frac{2}{2-r}$. Therefore for $r \geq 2d \geq 0$ the function f is increasing in total quantity Q for any $Q(x) \in \mathcal{X}$. It is easy to verify that $Q(x) \in \mathcal{X}$ for any $x \in \mathcal{X}$. By Lemma C.7, $\arg\max_{x\in\mathcal{X}^{\mathcal{K}}} f(Q(x)) = \overline{x}^{\mathcal{K}}$. Moreover, by Lemma C.8, when $r \geq 2d$, then $\arg\min_{x\in\mathcal{X}^{\mathcal{K}}} xR_F(x) = \overline{x}^{\mathcal{K}}$. We conclude that $\arg\max_{x\in\mathcal{X}^{\mathcal{K}}} W(x) = \overline{x}^{\mathcal{K}}$, for $r \geq 2d$.

Suppose that instead $r < 2d$. In this case $\frac{1}{1-d} > \frac{2}{2-r}$, hence for all $x \in \mathcal{X}$ it is the case that $R_F(x) > 0$ and $W(x) = a_0 + a_1 x + a_2 x^2$, where

$$a_0 := \frac{3-r}{2(2-r)^2}; \quad a_1 := 1 - \frac{(3-r)(1-d)}{(2-r)^2}; \text{ and}$$
$$a_2 := \frac{(d+1-r)^2 + (2-r)(1-d)(3-d) - (2-r)^3}{2(2-r)^2}.$$

There are three cases, depending on the sign of a_2.

(i) Consider the case $a_2 = 0$. This happens if and only if

$$d = d^*(r) := 1 - \frac{(2-r)\sqrt{(2-r)^2-1}}{3-r}.$$

Note that (i) $d^*(r)$ is strictly increasing over the interval $[0,1]$, (ii) $d^*(1) = 1$, and (iii) $2d^*(r) = r \iff r = 1/3$. So $a_2 = 0$ requires that $r \in (\frac{1}{3}, 1]$ and $d = d^*(r)$. Replacing d with $d^*(r)$ in a_1 gives:

$$a_1 = 1 - \sqrt{1 - \frac{1}{(2-r)^2}} > 0.$$

Therefore $\arg\max_{x \in \mathcal{X}^{\mathcal{K}}} W(x) = \overline{x}^{\mathcal{K}}$.

(ii) Suppose that $a_2 > 0$. Note that (i) $a_2 > 0$ if and only if $d < d^*(r)$, and (ii) for $a_2 > 0$ function $W(x)$ is convex and reaches a minimum at $\frac{-a_1}{a_2}$. We distinguish two cases.

(a) If $r \leq 1$, then $a_1 \geq 0$. To see this, note that (i) a_1 is increasing in d, so a_1 for $d = \frac{r}{2}$ is strictly smaller than for any $d \in (\frac{r}{2}, d^*(r))$, and (ii) evaluating a_1 for $d = \frac{r}{2}$, gives

$$\frac{1-r}{2(2-r)} \geq 0.$$

As $\frac{-a_1}{a_2} \leq 0$, then $\arg\max_{x \in \mathcal{X}^{\mathcal{K}}} W(x) = \overline{x}^{\mathcal{K}}$.

(b) Let $r > 1$. As $r < 2d$, Corollary C.1 and Lemma C.9 together ensure that $\underline{x}^{\mathcal{K}} = x^C$. Now,

$$x^C - \frac{-a_1}{2a_2} = \frac{A(r,d)}{B(r,d)},$$

where

$$A(r,d) := \frac{(2-r)(d+1-r)}{(3-r-d)},$$
$$B(r,d) := (1-r+d)^2 + (2-r)(1-d)(3-d) - (2-r)^3.$$

Clearly $A(r,d) > 0$ for the relevant values of r and d. We show that $B(r,d) > 0$. To see this, note that (i) $B(r,d)$ is convex in d, with minimum at $d = 1$, therefore $B(r,d)$ decreasing in $d \in [0,1]$, and (ii) $B(r,1) = (2-r)^2(r-1) > 0$ for $r > 1$. Again, as $W(x)$ is increasing in x over plausible values, it is maximized by $\overline{x}^{\mathcal{K}}$.

(iii) Finally, suppose that $a_2 < 0$. In this region $W(\cdot)$ is concave and reaches a maximum at $\frac{-a_1}{2a_2}$. As parameters satisfy $\min\{2d, 1\} > r$, $d > d^*(r)$, to conclude the proof it suffices to show that

$$\frac{-a_1}{2a_2} \geq \overline{x}^{\mathcal{K}}, \quad \forall r < 1 \text{ and } \forall d > d^{**}(r),$$

where

$$d^{**}(r) := \begin{cases} 0 & \text{if } r \leq 0, \\ \frac{r}{2} & \text{if } 0 < r \leq \frac{1}{3}, \\ d^*(r) & \text{if } r > \frac{1}{3}. \end{cases}$$

Simple algebra shows that $2d < r^{**}(d)$ for all $d \in [0,1]$, hence $r < 2d$ ensures $r < r^{**}(d)$. Corollary C.1 and Lemma C.9 together thus ensure that

$$\overline{x}^{\mathcal{K}} = \frac{(2-r)^2}{(3-r-d)\left((2-r)^2 - 2(1-d)^2\right)}.$$

Therefore

$$\frac{-a_1}{2a_2} - \overline{x}^{\mathcal{K}} = \frac{F(r,d)}{D(r,d)E(r,d)(3-r-d)}$$

where

$$D(r,d) := (2-r)^3 - (1-r+d)^2 - (2-r)(1-d)(3-d);$$
$$E(r,d) := (2-r)^2 - 2(1-d)^2;$$
$$F(r,d) := (2-r)(3-r-d)(3r - 3r^2 + r^3 - 2d + rd - r^2 d + 2d^2)$$
$$+ (2 - 4r + r^2 + 4d - 2d^2)((2-r)^3 - (2-r)(1-d)(3-d) - (1-r+d)^2).$$

Clearly $(3 - r - d) > 0$ for all (r, d) such that $r < 1$, and $d > d^{**}(r)$. We show next that for these parameter values $D > 0$ and $E > 0$. Note that both D and E are concave functions of d, and they both reach a maximum at $d = 1$. We conclude that both D and E are increasing functions of d for all $d \in [0, 1]$. We consider, in turn, cases $r \leq 0$, $r \in (0, \frac{1}{3}]$ and $r \in (\frac{1}{3}, 1)$.

(a) If $r \leq 0$, then $d^{**}(r) = 0$. We just established that $D(r, d) \geq D(r, 0)$ for all $d \in [0, 1]$. As $D_1(r, 0) < 0$, then $D(r, d) \geq D(r, 0) \geq D(0, 0) = 1$. Similarly, $E(r, d) \geq E(r, 0)$ for all $d \in [0, 1]$. As $E_1 < 0$, then $E(r, d) \geq E(r, 0) \geq E(0, 0) = 2$.

(b) If $r \in (0, \frac{1}{3}]$, then $d^{**}(r) = \frac{r}{2}$, and $D(r, d) \geq D(r, \frac{r}{2}) = 1/4(2 - r)^2(1 - 3r) \geq 0$, while $E(r, d) \geq E(r, \frac{r}{2}) = 1/2(2 - r)^2 > 0$.

(c) If $r \in (\frac{1}{3}, 1)$: then $d^{**}(r) = d^*(r)$, and $D(r, d) \geq D(r, d^*(r)) = 0$, while $E(r, d) \geq E(r, d^*(r)) = \frac{(2-r)^2(r+1)}{3-r} > 0$.

In the rest of the proof we show that $F(r, d) \geq 0$ for all $r \leq 1$ and $d \in [0, 1]$.

For any $d \in [0, 1]$, the function $F(r, d)$ is a 4th degree polynomial function of r. To prove that it is non-negative for all $r \leq 1$ and $d \in [0, 1]$, it suffices to show that for all $d \in [0, 1]$: (i) $F(1, d) > 0$ and (ii) $F(\cdot, d)$ does not have any real roots in $(-\infty, 1)$. To prove the first claim, note that:

$$F(1, d) = 4 - 17d + 28d^2 - 18d^3 + 4d^4.$$

All four roots of this polynomial are complex, and, for example, $F(1, 1) = 1 > 0$. Therefore $F(1, d) > 0$ for all $d \in [0, 1]$.

To prove the second claim, we use Sturm's theorem. For any $d \in [0, 1]$, let: $p_0(r) := F(r, d)$, $p_1(r) := F_1(r, d)$, $p_2(r) = -\operatorname{rem}(p_0(r), p_1(r))$, $p_3(r) = -\operatorname{rem}(p_1(r), p_2(r))$ and $p_4(r) =$

$-\operatorname{rem}(p_3(r), p_4(r))$, where $\operatorname{rem}(a, b)$ is the remainder of the Euclidean division of a by b. So

$$p_1(r) = 4r^3 + 6r^2d^2 - 15r^2d - 15r^2 - 24rd^2 + 60rd + 12r - 2d^4 + 10d^3 + 6d^2 - 46d;$$

$$p_2(r) = -\frac{1}{16}(1-d)^2 \begin{pmatrix} 32 + 60r - 27r^2 - 102d - 72rd + 36r^2d \\ +76d^2 + 24rd^2 - 12r^2d^2 - 22d^3 + 4d^4 \end{pmatrix};$$

$$p_3(r) = \frac{32(1-d)^2}{3(2d-3)^4} \begin{pmatrix} 16rd^4 - 104rd^3 + 212rd^2 - 126rd - 24r \\ -52d^4 + 314d^3 - 600d^2 + 335d + 55 \end{pmatrix};$$

$$p_4(r) = \frac{(1-d)^4(2d-3)^4\left(64d^6 - 672d^5 + 2340d^4 - 2984d^3 + 252d^2 + 1560d + 197\right)}{64\left(8d^4 - 52d^3 + 106d^2 - 63d - 12\right)^2}.$$

Sturm's theorem ensures that the number of real roots of $F(\cdot, d)$ in $(-\infty, 1]$ is equal to $V(-\infty) - V(1)$, where $V(r)$ denote the number of sign changes at r. We prove below that $V(-\infty) = V(1) = 2$, so that indeed the theorem ensures that $F(\cdot, d)$ does not have any real roots in $(-\infty, 1)$.

First, we establish that $V(-\infty) = 2$. To see this note that, at $r \to -\infty$ the sign of the polynomial are

- positive for $p_0(r)$ (a 4th degree polynomial with leading coefficient 1);
- negative for $p_1(r)$ (a 3rd degree polynomial with leading coefficient 4);
- positive for $p_2(r)$ (a 2nd degree polynomial with leading coefficient $\frac{3}{4}(1-d)^2\left(\frac{3}{2}-d\right)^2 > 0$);
- positive for $p_3(r)$ (a linear function with negative slope for all $d \in [0, 1]$).
- positive for $p_4(r)$ (a positive constant).

The number of sign changes is therefore 2. Next, we establish that $V(1) = 2$. To see this note that, at $r = 1$ the sign of the polynomial are

- positive for $p_0(r)$, $p_3(r)$ and $p_4(r)$;
- positive if $d < a_1$ and negative if $d > a_1$, where $a_1 \approx 0.278$ for $p_1(r)$;
- is negative if $d < a_2$ and positive if $d > a_2$, where $a_2 \approx 0.845$ for $p_2(r)$.[26]

For any $d \in [0, 1]$, the number of sign changes is indeed 2. ■

[26] The exact values of a_1 and a_2 do not change the conclusions.