Overbooking *

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Abstract

We consider optimal pricing policies for airlines when passengers are uncertain at the time of ticketing of their eventual willingness to pay for air travel. Auctions at the time of departure efficiently allocate space and a profit maximizing airline can capitalize on these gains by overbooking flights and repurchasing excess tickets from those passengers whose realized value is low. Nevertheless profit maximization entails distortions away from the efficient allocation. Under standard regularity conditions we show that the optimal mechanism can be implemented by a modified double auction. In order to encourage early booking, passengers who purchase late are disadvantaged. In order to capture the information rents of passengers with high expected values, ticket repurchases at the time of departure are at a subsidized price, sometimes leading to unused capacity.

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1 Introduction

Overselling limited seating space is standard practice among airlines. According to US Department of Transportation reports, between the months of April and June of 2011, about one per cent of passengers ticketed on a US domestic flight was denied boarding on that flight. Mitigating the potential inconvenience for passengers, airlines typically look for volunteers, raising the level of compensation offered until enough passengers have been found willing to delay their travel. As a result, only about one per cent of passengers who are denied boarding on oversold flights are bumped involuntarily.\(^1\)

Airlines typically explain their practices as motivated by the fact that a certain number of passengers can be expected not to show up (on time) for a flight and overbooking capitalizes on slack capacity, improving efficiency. From this perspective, how much an airline should overbook is chiefly a statistical question, one which requires balancing the value of unused capacity against costs associated with overbooked flights, including inconvenience to passengers and possible compensation. This is also the view taken by a large literature in operations research (early contributions include Beckmann (1958) and Rothstein (1971)).\(^2\)

In this paper, we explore a different but complementary rationale for overbooking. This rationale is based on price discrimination among passengers who face uncertainty about their eventual value for being seated on the flight. Unlike the operations-based perspective on overbooking, passenger incentives—both to purchase tickets and to give up their seats for compensation—are central to our theory. We focus on a setting where ticket sales are not needed to achieve an efficient allocation, but where they instead play a role in extracting surplus from passengers who are uncertain about their values for flying in advance of the flight. Of course, one expects that, in practice, the traditional rationale for overbooking also plays a role. Our modeling choice is motivated by simplicity and a desire to understand which properties of pricing policies arise from the surplus-extraction motive.

We consider an airline which can offer homogeneous tickets at a single price, akin to offering only a single fare class. We show that the airline may profit from selling more tickets than available capacity. Rather than indiscriminately canceling tickets when a flight is overbooked, i.e. involuntary bumping, the airline always finds it profitable to

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\(^1\)Attempts to elicit volunteers are becoming more sophisticated. For instance, Delta Airlines has recently implemented a system in which passengers indicate, at the time of check-in, their willingness to take a later flight. They do so by nominating the value of travel vouchers that they would be willing to receive to delay their travel.

\(^2\)There are a few recent exceptions. For example in Gallego and Sahin (2010) the airline sells options to fly and passengers whose values turn out out to be low take the refund rather than fly, but in contrast to our assumptions, in their model the refund is independent of the demand.
treat tickets as conferring a right to a seat. The airline must then repurchase these rights. Although airlines in practice withhold the authority to bump even unwilling passengers, subject to rules surrounding compensation, our finding does seem consistent with the status quo in the U.S., where involuntary bumping occurs only rarely.

Procuring oversold capacity via auction offers airlines the chance to improve ex-post efficiency of the seating allocation. The choice of reallocation mechanism also affects the demand for tickets. Compensation that passengers anticipate in the event of overbooking improves the value of obtaining a ticket in advance of the flight. Therefore, the design of mechanisms for determining the final seat allocation cannot be separated from the choice of ticket pricing policies. We shed light on this connection.

In our theory, the passengers who purchase tickets in advance of the flight will be those who have sufficiently favorable information about their value for flying. Among the ticket purchasers, those with the least favorable information will be at the margin. Such passengers are, of course, more likely to benefit from the compensation offered at the date of travel. The possibility of future compensation allows the airline to raise the price of the ticket without affecting the payoffs of the marginal ticket holders, and therefore without reducing the number of tickets sold. In so doing, it reduces the consumer surplus of the infra-marginal ticket holders. This is simply because infra-marginal ticket holders pay the higher ticket price but are less likely to benefit from the compensation for giving up their seat. In this way, the promise of future compensation is a tool for the airline to limit consumer surplus.

While the above observations seem consistent with practice, our theory also suggests possible ways that airlines could tailor seat reallocation to improve profits. For example, the airline may profit from repurchasing seating rights even when capacity constraints do not bind. Whether such policies might improve profits does depend, however, on the distribution of passenger values (and there may, perhaps, be other reasons airlines would find them undesirable).

Airlines can also improve profits by reallocating seating rights to passengers who do not purchase tickets in advance, either because they anticipate a low value for flying or because they are “out of the market”. A passenger being out of the market simply captures the possibility that their need for travel is yet to arise. We show that the airline profits by giving such passengers the opportunity to fly, but that doing so improves the value of the option associated with not purchasing a ticket. It therefore reduces the demand for tickets. This consideration therefore encourages the airline to charge higher prices to unticketed passengers, an observation which seems to help explain the substantially higher

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3The U.S. Department of Transportation sets out the rules for compensation in case of involuntary bumping in 'Fly-Rights: A Consumer Guide to Air Travel'. Compensation for involuntary bumping is mandatory, and may be as high as US$1,300.
ticket prices often observed in the two or three weeks immediately preceding a flight (see McAfee and te Velde (2006) and Lazarev (2012) for these pricing patterns).

We are able to quantify the airline’s incentive to allocate rationed space between ticket holders and non-ticket holders using a version of the familiar virtual surplus measure. These formulas show how allocation decisions affect revenues directly through departure-time transfers but also indirectly through their affect on passengers’ willingness to pay for tickets in advance. Under appropriate regularity conditions, we show that the optimal mechanism can be implemented using a modified double auction.

Finally, our analysis offers a way to evaluate the efficiency and welfare consequences of overbooking and seat reallocation. This kind of analysis may be important for policy makers considering imposing possible restrictions on overbooking. We argue that overbooking and subsequent reallocation is potentially efficiency enhancing (although we provide no general results about the direction of the effect).

The rest of the paper unfolds as follows. After discussing related literature in the rest of this section, we introduce a model in Section 2. Section 3 introduces the mechanisms where a single kind of ticket is sold before the date of travel as described above. To illustrate how this mechanism operates, we provide a stylized example. In Section 4 we derive the optimal transfers and allocation rules for these pricing mechanisms and show how to implement the optimal mechanism by a modified double-auction. In Appendixes we provide proofs of all results and derive the unrestricted optimal mechanism and show how our analysis can be extended to multiple fare classes.

Related literature

Julian Simon first suggested auction mechanisms as a response to overbooking in the airline industry in the mid-1960s (see Simon (1994)). These suggestions were implemented by the industry in the US from 1978, with volunteers being selected on the basis of willingness to give up their seat, rather than arbitrarily. In another notable contribution, Vickrey (1972) proposed the use of an efficient auction to resolve the allocation of seats in the event of overbooking. Vickrey proposed extending this mechanism to include dynamic flexible pricing schemes, and speculated about many of the issues that we address here, although without a formal model.

Our paper contributes also to the more recent literature on selling to consumers who

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4In the U.S. overbooking has long been accepted practice, but policy is still evolving in other jurisdictions. In the Philippines, the regulator recently proposed a ban on overbooking, although policy has now evolved towards permitting and regulating it (‘Passenger bill of rights out soon’, www.philstar.com, October 27, 2012).

5Simon (1994) notes that, before 1978, United Airlines, for instance, followed a practice of bumping “old people and armed services personnel, on the assumption that they would be least likely to complain.”
learn about their valuations over time. An important question we are concerned with is: When should the firm allocate available capacity, before or after the buyer has learned his value? Biehl (2001), Deb (2011), Dana (1998), Möller and Watanabe (2010) and Nocke et al. (2011) provide models where sellers choose to discriminate between consumers based on their prior information, for instance by offering “advanced purchase discounts”. DeGraba (1995) and Courty (2003) study related models where there is no role for inter-temporal discrimination due to buyers lacking private information at the initial stage. The key distinction relative to our work is that capacity constraints in these papers are either respected (e.g., DeGraba and Möller and Watanabe)—tickets sold do not exceed the available capacity—or completely absent (e.g., Biehl, Courty, Deb, and Nocke et al.). We instead demonstrate how a seller (airline) may find it optimal to over-allocate capacity and then use an appropriate mechanism for ensuring capacity constraints are respected on the date of consumption.

There is also a recent branch of literature in revenue management and marketing that considers buyers with evolving valuations and shows that advance selling with partial refunds (or selling options to fly) ensures higher revenue than selling non-refundable tickets (Gallego and Sahin, 2010; Gallego et al., 2008) and discounts for advance selling may increase revenues (Shugan and Xie, 2000; Xie and Shugan, 2001; Gallego and Sahin, 2010; Gallego et al., 2008). However, while these papers include more institutional details than ours, they assume that refund for cancellations is constant, which seriously limits the airlines choices. In particular, in our model the refund as well as whether the passenger will fly or not will be determined as a comparison with other passenger values at a re-allocation auction before the departure.

In work independent of our own, Fu et al. (2012) also study a setting where an airline can over-allocate capacity, and then either repurchase it or randomly select passengers not to fly. Unlike our work, the airline can only allocate seats to passengers who buy tickets in advance of the flight. Overselling of tickets therefore plays the role of improving efficiency by helping the airline match the highest value passengers to the available capacity. In contrast, the airline in our model can achieve full efficiency without advance ticket sales.

The restriction to homogeneous tickets means the airline loses some of its ability to screen the initial information of passengers. To screen passengers, the airline may strictly prefer to set ticket prices that are unattractive for some passengers with unfavorable beliefs about their values for flying. The airline therefore balances trade-offs which are absent from most of the literature on dynamic mechanism design which imposes no restriction on the mechanism (e.g., Baron and Besanko (1984), Courty and Li (2000), Battaglini (2005), Eso and Szentes (2007) and Pavan et al. (2013)). For most of the papers in that literature, the principal (most often a seller) optimally contracts with all agents at the
time of commitment to the mechanism. An important exception is Deb and Said (2013). Unlike the rest of the literature, the seller in their model cannot commit to the contracts it offers at later dates. Exclusion of buyers is important as it dictates the composition of buyers available to sign later contracts, effectively providing a form of commitment to less generous contracts at later dates.

Selling tickets in our model also provides a way for the airline to distinguish timely arrivers from those who arrive to the market late. Our paper is thus related to a literature studying buyers who arrive over time but who face no uncertainty about their future valuations. Examples include Gershkov and Moldovanu (2009, 2010), Board and Skrzypacz (2010) and Said (2012) (see, Bergemann and Said (2010) for an overview as well as McAfee and te Velde (2006) for a review focused on applications to the airline industry). Further examples include work by Pai and Vohra (2013) and Mierendorff (2011), who provide models where buyers arrive over time and have private deadlines. Mierendorff shows that the optimal mechanism may feature a handicap auction which favors the early arriver, in order that the early arriver truthfully disclose a late deadline. In contrast to all of these papers, we study an environment where buyers are uncertain about their future values for the good. The treatment of a setting with both dynamic arrivals and valuation uncertainty builds on recent work by Garrett (2010, 2013).

2 Model

A risk-neutral monopolist airline is selling $m$ seats on a flight that departs at date 1. There are $n$ risk-neutral potential passengers who are ex ante anonymous and symmetric, where $n \geq m$. A passenger may enter the market at date 0 or only at date 1. A passenger arriving at date 0 receives partial information about his value for flying at date 1, $v_i$, which is captured by a signal $\theta_i$. Vectors of values are denoted in bold font, i.e. $v = (v_i)_{i=1}^n$, with $v_{-i'} = (v_i)_{i \neq i'}$ for any individual passenger $i'$. Both a passenger’s time of arrival and information about his value for flying are determined independently of the other passengers’ realizations and are his private information.

The signals $\tilde{\theta}_i$ are drawn from a distribution with a CDF $F$ whose support is $\Theta = [0,1] \cup \{\emptyset\}$. The signal $\theta_i = \emptyset$ indicates that the passenger is out of the market, and unavailable for ticket purchases at date 0. For notational convenience we will adopt the convention that $\emptyset < 0$. A signal $\theta_i \in [0,1]$ indicates that the passenger enters the market at date 0 and we assume that $F$ admits a density $f$ over that range and include

6A further exception is Nocke et al. (2011), where the seller optimally delays contracting to some buyers, but where the same profits can be obtained by inducing participation by all.

7See Lazarev (2012) for a recent empirical study of dynamic pricing in the airline industry.

8Throughout, random variables are denoted using tildes.
the possibility of an atom at $\emptyset$. Abusing notation slightly, $F(\theta_i|S)$ denotes conditional distribution of $\theta_i$, conditional on event $S$.

Conditional on his signal $\theta_i$, a passenger $i$’s (non-negative) eventual willingness to pay $\tilde{v}_i$ is distributed according to the CDF $G(v_i|\theta_i)$, where $G(\cdot|\cdot)$ is continuously differentiable, and where the density is denoted $g(v_i|\theta_i)$. The support of the marginal distribution of $\tilde{v}_i$ is $[\underline{v}, \overline{v}]$. (The support of the distribution of $\tilde{v}_i$ conditional on a particular realization $\theta_i$ of the signal may be a strict subset of $[\underline{v}, \overline{v}]$.) Abusing notation, we also let $G(v|S)$ and $G(v_{-\ell}|S_{-\ell})$ denote the joint distributions of passenger values conditional on the events $(\theta_i)_{i=1}^n \in S \subset \Theta^n$ and $(\theta_i)_{i \neq \ell} \in S_{-\ell} \subset \Theta^{n-1}$.

Signals are ordered in the sense of first-order stochastic dominance, so, for all $v_i$, $G(v_i|\theta_i)$ is nonincreasing in $\theta_i$ over $[0, 1]$ (at this point, we impose no additional restrictions on $G(\cdot|\emptyset)$). To simplify some of the analysis, we further impose the weak restriction that there exists $k > 0$ such that, for all $x$, $G(k\theta_i + x|\theta_i)$ is nondecreasing in $\theta_i$.\(^9\)

The airline may sell tickets at date 0 and reallocate capacity (including to unticketed passengers) at date 1. We describe the mechanisms that the airline can use below. There is no discounting. Passengers realize utility $v_i$ if they are allocated a seat and a utility zero otherwise, less any payments made to the airline across the two periods. So a passenger whose (possibly negative) total payment to the airline is $\rho$ earns utility $v_i - \rho$ if he is seated and $-\rho$ if he is not. Given a fixed number of available seats $m$ and a zero cost of seating a passenger in an available seat, the airline maximizes the expected total payment by passengers.

### 3 Pricing Mechanisms

We consider pricing mechanisms $\Omega$ of the following form. At date 0, the airline sets a price $p$ for tickets, and also commits to a family of re-allocation mechanisms\(^10\) to be used at date 1 depending on the number of tickets sold.\(^11\) In particular, the mechanism will be used to determine which passengers will sell back their tickets in the event of overbooking, to broker the possible transfer of seating rights from the date-0 purchasers to those unticketed passengers who are willing to pay to fly, and possibly to sell additional seats to passengers who wish to purchase tickets at date 1. Throughout, we restrict attention to mechanisms

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\(^9\)This allows us to rely directly on arguments in Pavan et al. (2013); see the proof of Lemma 1. In words, the condition says that there exists $k$ such that, for any $x$, the probability that a passenger’s value for flying exceeds his signal $\theta_i$ by more than $k\theta_i + x$, is nonincreasing in his signal. (We are grateful to Phil Reny for suggesting this interpretation.)

\(^10\)The way we model assignment of future service is analogous to Hinnosaar (2014), but in his model agents are fully informed at arrival and there is no overbooking.

\(^11\)We assume full commitment. Deb and Said (2013) show in a similar framework that optimal dynamic rationing can provide partial commitment power even for a seller who cannot explicitly commit to future contractual terms.
that treat passengers symmetrically, and, given that mechanisms may be random, this comes with no loss to the airline.

**Feasibility and incentive constraints.** The mechanism \( \Omega \) must satisfy familiar feasibility and incentive constraints. First, at most \( m \) passengers can be seated on the plane. Second, the mechanism must be implementable in perfect Bayesian equilibrium: whether or not a passenger purchases a ticket, his continuation play at date 1 must be optimal given updated beliefs. Our assumption is that at date 1, absent any information provided by the airline, passengers act without knowing how many other tickets have been sold and the signals or values of the other passengers. It will turn out, however, that compared to Bayesian incentive compatible mechanisms there is no loss in revenue in using mechanisms that inform all passengers of the total number of tickets sold and that ask passengers to play dominant strategies at date 1 (see Lemma 2 below). At date 0, a passenger’s decision whether or not to purchase a ticket must be optimal given the anticipated continuation play.

Our treatment of participation constraints is familiar from the dynamic mechanism design literature. Tickets are a contractual commitment for both the airline and passengers, so we treat ticketed passengers as required to participate at date 1. However, we will show that, by designing ticket prices and transfers appropriately, the airline need not rely on the passengers’ commitments.\(^{12}\) For unticketed passengers, there is no contractual commitment (indeed, ticket sales are the only contract formed at date zero and ticket prices are the only date-zero transfers) and so these passengers are not compelled to participate at date 1.

**Additional constraints on the mechanism.** Our analysis focuses on a restricted class of mechanisms. The first restriction, already implicit in the discussion above, is that the airline is permitted to offer tickets for a single fare class with a single price \( p \).

Intuitively, selling tickets is a way of dividing customers among those who are present at date zero and have certain beliefs over their willingness to pay, and those who do not. When the set of possible customer date-zero beliefs is sufficiently rich, a finer partition will often be optimal, as could be achieved by offering multiple fare classes, each with a different set of terms.\(^{13}\) However, for the mechanisms we are considering, the airline has no way to distinguish between the different passengers who request a ticket at date zero.

The reasons for focusing on the sale of a single kind of ticket are twofold. First, we believe it may be unrealistic to expect airlines to distinguish finely between passenger

\(^{12}\)This is because the airline has a degree of freedom to shift payments across time. By increasing the price of the ticket and reducing any date-1 payment (equivalently, increasing any date-1 subsidy) to ticket holders, the airline can ensure that ticket holders find date-1 participation optimal even if they are not contractually bound.

\(^{13}\)Appendix B describes how our analysis can be extended to accommodate two or more fare classes, with different prices for different fares.
beliefs (in our model, the airline would find it profitable to offer a continuum of contracts with different terms; we derive the optimal mechanism with no restrictions on tickets in Appendix B).\textsuperscript{14} Second, while it may be realistic to consider a small number of fare classes, restricting attention to only one simplifies the analysis. In particular, it allows us to focus on the decision of passengers \textit{whether} to contract in advance of the flight, rather than the decision of \textit{which kind} of contract to purchase.

We assume that all passengers requesting a ticket at price $p$ receive one. This comes at no cost to the airline. Given that passengers are anonymous and ex-ante symmetric, the profit of any pricing mechanism which rations ticket sales can be attained by an alternative mechanism in which (i) a ticket is sold to every passenger who requests one, and (ii) ticketed passengers are randomly assigned either the allocations and payments associated with holding a ticket in the original mechanism with rationing, or the allocations and payments associated with not holding a ticket in that mechanism.

Our second restriction is that having a ticket cannot decrease the probability of being seated. In particular, if the mechanism seats an unticketed passenger with value $v_i$ at date 1, then it also seats the same passenger if he had instead purchased a ticket at date 0 (holding fixed the actions of the other passengers, and assuming that the passenger in question behaves optimally at date 1). In other words, obtaining a ticket at date 0 does not hurt a buyer’s probability of being seated at date 1.\textsuperscript{15} The idea here is that tickets are required to play the role with which we are familiar, i.e. providing privileged access to their holders. Our analysis focuses on the extent to which the airline profitably uses tickets in this role.\textsuperscript{16}

The above conditions allow us to focus attention on a class of mechanisms which is easy to analyze. Assuming for simplicity that passengers purchase tickets whenever indifferent, we show that such mechanisms have the following property.

**Lemma 1.** \textit{In any pricing mechanism satisfying the above conditions, there exists a threshold value $\theta^*$ such that passengers purchase tickets if and only if they arrive at date zero and have a signal no less than $\theta^*$.}

Excluding boundary cases in which all or no available passengers purchase tickets, $\theta^*$ must be a signal for which passengers are indifferent. We hence refer to this as the

\textsuperscript{14}Indeed, part of our objective is to explain how an airline can profit from overbooking with simple instruments which resemble those already used in the industry.

\textsuperscript{15}Note that this is different to assuming that ticketed passengers are seated ahead of unticketed passengers with the same value. We instead derive this as an implication of our model (under appropriate regularity conditions) in Proposition 2.

\textsuperscript{16}Note the assumption that tickets must not reduce the probability of being seated is the only restriction on the date-1 allocation mechanism which can affect airline profits. The airline thus chooses from a wide class of mechanisms for allocating seats at date 1, and can allocate seats to passengers not holding tickets. This means, for instance, that full efficiency is always attainable without selling tickets, simply by holding the appropriate auction at date 1.
“marginal signal” and to the passenger as the “marginal ticket holder”. Buyers with initial signals above $\theta^*$ anticipate higher values for flying in the sense of first-order stochastic dominance. Since obtaining a ticket increases a buyer’s chances of being seated, buyers with signals above $\theta^*$ strictly prefer purchasing tickets at date zero.

If the threshold signal $\theta^*$ is less than one, then (given that buyers draw signals symmetrically) there is a positive probability that the airline sells tickets to all $n$ passengers. If, in addition, $m < n$, then the airline practices overbooking. That is, there is a positive probability that more tickets are sold than available seats. Corollary 1 below will give sufficient conditions for $\theta^*$ to be less than one in the profit-maximizing pricing mechanism.

**The date 1 mechanism.** Before describing how the airline sets ticket prices, consider the mechanism it uses to allocate seats at date 1. We consider without loss of generality direct revelation mechanisms, where payments and allocations depend on which passengers hold tickets and their reported values for flying.

Let $s \in \{0, \ldots, n\}$ be the number of passengers holding tickets purchased at date 0. We will use subscripts $j$ to denote passengers that hold tickets at date 1 and subscripts $k$ to denote those who do not. Thus, we can order passengers so that $j = 1, \ldots, s$ hold tickets and the remaining passengers $k = s + 1, \ldots, n$ do not. We let $S^s$ denote this event, i.e.

$$S^s = \{\theta_j \geq \theta^*, j = 1, \ldots, s; \theta_k < \theta^*, k = s + 1, \ldots, n\}$$

and we denote by $S^s_i$, the corresponding event with passenger $i$ excluded.

At date 1, each passenger observes his realized willingness to pay as well as whether or not he is holding a ticket. A direct revelation mechanism is described by a collection of functions $(q, t)$, such that, for each $s \in \{0, \ldots, n\}$, $q^s(v) = (q^s_i(v))_{i=1}^n$ gives the probability that each passenger $i$ is seated while $t^s(v) = (t^s_i(v))_{i=1}^n$ gives the payment that each passenger $i$ makes to the airline at date 1.\(^\text{17}\) Conditional on $s$ tickets sold, we say passengers play the “$s$-mechanism”. To ease notation, and without loss of optimality, payments and probabilities of allocation are defined (unless otherwise specified) for all values in $[\underline{v}, \overline{v}]$, which includes values that may be inconsistent with equilibrium play.\(^\text{18}\)

Consider a passenger $i$ who truthfully reports his value $v_i$, while other passengers report $v_{-i}$, and suppose the total number of tickets sold is $s$. Disregarding the ticket price paid at date 0, if any, the payoff to passenger $i$ is $U^s_i(v) = q^s_i(v)v_i - t^s_i(v)$.

Let $\tilde{r}$ be the random variable representing the number of tickets purchased by passengers other than $i$, which thus follows a binomial distribution with parameters $(n - 1, 1 - \text{Pr}(\text{no ticket}))$.

\(^{17}\)While we allow for random allocations, there is no loss of generality in considering deterministic transfers due to the linearity of airline and passenger payoffs.

\(^{18}\)In particular, we require the mechanism to be defined if one of the bidders claims to have a value that is inconsistent with his decision whether to purchase a ticket, a possibility which arises because the support of $G(\cdot|\theta_i)$ may be a strict subset of $[\underline{v}, \overline{v}]$ for $\theta_i \in [0, 1]$.  

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The date 1 mechanism \((q, t)\) is Bayesian incentive compatible if, for each ticketed passenger \(j\), all \(v_j\) and all reports \(\hat{v}_j\),

\[
E \left[ U_j^{r+1}(v_j, \hat{v}_j) \right] \geq E \left[ q_j^{r+1}(\hat{v}_j, \hat{v}_j) v_j - t_j^{r+1}(\hat{v}_j, \hat{v}_j) \right]
\]

and, for each unticketed passenger \(k\), all values \(v_k\) and all reports \(\hat{v}_k\),

\[
E \left[ U_k^r(v_k, \hat{v}_k) \right] \geq E \left[ q_k^r(\hat{v}_k, \hat{v}_k) v_k - t_k^r(\hat{v}_k, \hat{v}_k) \right].
\]

The mechanism is \textit{ex-post} incentive compatible if, for all \(s\), all \(i\), \(v_i\), \(\hat{v}_i\) and \(v_{-i}\),

\[
U_s^r(v_i, v_{-i}) \geq q_i^s(\hat{v}_i, v_{-i}) v_i - t_i^s(\hat{v}_i, v_{-i}).
\]

Thus in an ex-post incentive-compatible mechanism the airline can publicly announce the number of ticket holders and truth-telling would remain ex-post optimal for all passengers.

The following lemma shows that any Bayesian incentive-compatible mechanism can be replaced by an ex-post incentive compatible mechanism that generates the same expected profit. The proof follows from essentially the same arguments made in Gershkov et al. (2012).

**Lemma 2.** There is no loss of optimality in restricting attention to ex-post incentive-compatible mechanisms, i.e. mechanisms in which the airline announces the number of tickets sold and truth-telling is ex-post optimal for all passengers.

**Ticket pricing.** The airline designs the date-1 mechanisms taking into account not only the revenues they generate, but also how they affect the passengers’ willingness to pay for a ticket at date 0. To formalize the tradeoffs we will need some notation.

We denote the expected utility of a ticketed passenger as a function of his date-0 signal, given the \(s\)-mechanism, by \(V^s(\cdot)\). For the unticketed passenger, it is \(V^s(\cdot)\). These expected payoffs are gross of the ticket price. Thus, for the signals of ticketed passengers \(\theta_j\) and of unticketed passengers \(\theta_k\),

\[
V^s(\theta_j) = E \left[ U_j^s(v)|\hat{\theta}_j = \theta_j, \hat{\theta}_{-j} \in S^s_{-j} \right] \quad \text{and} \quad V^s(\theta_k) = E \left[ U_k^s(v)|\hat{\theta}_k = \theta_k, \hat{\theta}_{-k} \in S^s_{-k} \right].
\]

The expected per passenger profit earned at date 1 from ticketed and unticketed passengers in the \(s\)-mechanism is given by

\[
\Pi^s = E \left[ t_j^s(v)|\hat{\theta} \in S^s \right], \quad \text{and} \quad \Pi^s = E \left[ t_k^s(v)|\hat{\theta} \in S^s \right].
\]

Note that \(\Pi^s\) and \(\overline{V^s}(\theta_j)\) are defined only for \(s > 0\), since when the number of ticketed passengers is \(s = 0\), the profit from and value for a ticketed passenger are irrelevant.
Similarly, $\Pi^s$ and $V^s(\theta_j)$ only for $s < n$, since when $s = n$, there are no unticketed passengers.

The expected payoff as of date zero to a passenger $i$ with signal $\theta_i$ who purchases a ticket at price $p$ is $E_p[\bar{V}^{\bar{r}+1}(\theta_i)] - p$, where the number of other ticketholders, $\bar{r}$, is determined according to the binomial distribution given above. The expected payoff to this passenger if not purchasing a ticket is $E_p[\bar{V}^{\bar{r}}(\theta_i)]$. In an optimal pricing mechanism, for any $\theta^* \leq 1$, the ticket price $p$ must satisfy

$$E_p[\bar{V}^{\bar{r}+1}(\theta^*)] - p = E_p[\bar{V}^{\bar{r}}(\theta^*)]. \tag{1}$$

Equation (1) states that the marginal ticketholder is indifferent between purchasing a ticket and not purchasing. When $\theta^* > 0$, given that signals are continuously distributed on $[0, 1]$, this condition is necessary for passengers to optimally follow the desired ticket purchasing strategy. If $\theta^* = 0$, then a range of ticket prices are consistent with incentive compatibility, however Eq. (1) specifies the airline’s uniquely profit-maximizing choice.\(^{19}\)

Equation (1) allows us to express the ticket price in terms of the marginal ticket-buying signal. For any threshold signal $\theta^*$

$$p = \sum_{r=0}^{n-1} \binom{n-1}{r} (1 - F(\theta^*))^r F(\theta^*)^{n-1-r} [\bar{V}^{\bar{r}+1}(\theta^*) - \bar{V}^{\bar{r}}(\theta^*)]. \tag{2}$$

The airline’s expected profit is therefore

$$\pi = n (1 - F(\theta^*)) p + E_s[s\Pi^s + (n - s)\Pi^s],$$

where the number of ticket purchasers, $\tilde{s}$, has binomial distribution with parameters $(n, 1-F(\theta^*))$. After substituting Eq. (2) and rearranging, expected profit can be expressed in terms of the date 1 mechanisms and a threshold ticket-buying signal $\theta^*$;\(^{20}\)

$$\sum_{s=0}^{n} \phi^s(\theta^*) \left[ s \left( \Pi^s + V^s(\theta^*) \right) + (n-s) \left( \Pi^s - \frac{1 - F(\theta^*)}{F(\theta^*)} V^s(\theta^*) \right) \right], \tag{3}$$

where the probability of selling exactly $s$ tickets, $\phi^s(\theta^*)$, is defined by

$$\phi^s(\theta^*) = \binom{n}{s} F(\theta^*)^{n-s} (1 - F(\theta^*))^s.$$

From Equation (3) we can directly see several implications for our analysis. First, it

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\(^{19}\)A lower choice of $p$ is consistent with $\theta^* = 0$, but the airline can then increase profits by raising $p$ without affecting passenger incentives.

\(^{20}\)See Appendix A for a derivation.
suggests that we may be able to solve for the optimal reallocation mechanism for each $s$ separately. We take advantage of this separability to characterize optimal pricing mechanisms in what follows. Second, it emphasizes the relevant factors in revenue maximization. Of course, it includes revenue from selling seats to passengers in the day of departure, $\Pi^s$, as well as (what will be nonpositive) revenue from buying back seats allocated to ticketed passengers, $\overline{\Pi}$. Additionally, it includes the surplus that the marginal ticketed passenger expects to receive, $\overline{V}^s$, and the value of the surplus of marginal unticketed passenger, $\overline{V}^s$, which affects the revenue negatively, since it decreases the motivation to purchase tickets in advance. The surplus term $\overline{V}^s$ is weighted by $\frac{1 - F(\theta^*)}{F(\theta^*)}$ reflecting the relative probability of a passenger being ticketed rather than unticketed; i.e., the importance of $\overline{V}^s$ in determining profits is increasing in the relative probability that a ticket is sold. Notice that the surplus terms $\overline{V}^s$ and $\overline{V}^s$ are determined for marginal consumers, because they are the ones determining the ticket price, whereas revenue terms are averages over the infra-marginal consumers. In the next subsection we illustrate trade-offs and implications arising from this by focusing on a simple special case.

**Illustrative Example**

In this section we will illustrate the main trade-offs faced by the airline in deciding the price for the ticket, how seats will be allocated and whether to overbook. We do so by means of a simple discrete example with two passengers and one seat to be allocated.

As the example will show, in general the airline will find it optimal to sell tickets in advance and potentially re-allocate the tickets prior to the departure. It also illustrates the new sources of distortions. First, the discussion of repurchase mechanism (or $s = 2$ case) illustrates why it is sometimes optimal to leave seats empty even if there are consumers who have paid for the ticket and have positive value from flying. Second, the discussion of the spot mechanism ($s = 0$) shows why the optimal mechanism allocation for the unticketed passengers even more than a standard spot auction. Third, the discussion of the re-allocation mechanism ($s = 1$) argues that the same forces are present generally and the sign of the distortion depends on the magnitudes of both forces. Finally, we discuss the advantages of early ticket sales.

In this example, we assume that the buyer in our example either arrives at date zero with a signal (his “type”) in $\{\theta, \overline{\theta}\}$, $\theta < \overline{\theta}$, or at date one, with type $\emptyset$. The probabilities of each type satisfy $f(\theta) + f(\overline{\theta}) + f(\emptyset) = 1$, with $f(\theta), f(\overline{\theta}), f(\emptyset) \in (0, 1)$.

---

\[ (1 - F(\theta^*)) \left( \overline{\Pi}^1 + \overline{V}^1(\theta^*) \right) + F(\theta^*) \left( \overline{\Pi}^0 - \frac{1 - F(\theta^*)}{F(\theta^*)} \overline{V}^0(\theta^*) \right) = (1 - F(\theta^*)) (\overline{\Pi}^1 + p) + F(\theta^*) \overline{\Pi}^0. \]
In the second period, each passenger $i$ realizes willingness to pay $v_i$ in $\{v, \overline{v}\}$ where $0 < v < \overline{v}$. To demonstrate the key ideas, the signal $\overline{\theta}$ represents an inflexible passenger: with probability 1 his willingness to pay will be $\overline{v}$. On the other hand, $\theta$ and $\emptyset$ represent passengers who are uncertain about their eventual willingness to pay: $g(\overline{v}|\overline{\theta}), g(v|\emptyset) \in (0,1)$. We further assume that

$$\frac{f(\overline{\theta})}{f(\theta)} < \frac{g(\overline{v}|\emptyset)}{g(v|\emptyset)} + \frac{1 - f(\emptyset)}{f(\emptyset)} \frac{g(\overline{v}|\theta)}{g(v|\emptyset)}.$$  \hspace{1cm} (4)

It turns out that this condition simplifies the analysis by allowing us to ignore one of the incentive constraints; namely, that a passenger with signal $\overline{\theta}$ does not want to “hide” and participate in the mechanism only at date 1.

We focus on optimal date-1 mechanisms when ticket prices are low enough that all early arriving passengers purchase tickets.\(^{22}\) In this particular example, such mechanisms are optimal, not only given the special constraints that we impose on the mechanism as described above, but also when there are no such restrictions.\(^{23}\) The Appendix gives a detailed proof of this last observation, which follows only because we are assuming $g(\overline{v}|\overline{\theta}) = 1$.\(^{24}\)

It turns out that we can restrict attention to mechanisms whose final allocation has “no distortion at the top”. That is, if there is at least one passenger with the high value $\overline{v}$, then, regardless of how many tickets were purchased in the first period, the seat will not be left empty and at least one of the high-value passengers will be seated. For example if both high-value passengers are ticketed passengers, only one of them will be seated (we suggest below that the airline may favor committing to compensate the passenger who is not seated). Likewise, if neither is holding a ticket, then the seat will be sold to one of them. In both of these cases, the tie-breaking can be arbitrary. When only one of the two high-value passengers has a ticket, there is no loss in assuming that it is the ticketholder who keeps his seat.

To see why it is never optimal to leave the seat empty when there are high-value passengers, note that excluding a high-value passenger is dominated by seating him (when possible) and extracting all of his rents by charging him (an additional) $\overline{v}$ for the seat.

\(^{22}\)It is easy to see that under the assumptions made above, case when the optimal ticket price excludes passengers with low signal, the dynamic mechanism cannot give higher revenue than simple spot auction in date 1. Therefore this case would be trivial. Of course, to finalize the analysis, we need to compare revenues from these two cases.

\(^{23}\)The reason for this finding relates to the fact that, in this example, the allocation rule need not distinguish between low-value passengers with different date-zero signals. In particular, any low-value passenger who arrives at date zero must have received signal $\overline{\theta}$.

\(^{24}\)Our arguments below focus on optimality within the restricted class of mechanisms. While these arguments are focused more on the intuition, the Appendix offers a compact stand-alone proof of the optimality of the mechanism derived here.
in the second period. It clearly increases profits for the airline and, since it leaves all passengers’ first-period expected payoffs unchanged, it remains incentive-compatible.

Thus, in this example, the problem reduces to deciding how to treat passengers when both have low realized values ($v$) depending on which of the two (if any) purchased tickets in the first period. Let’s begin by considering the case in which both passengers have low values and are holding tickets so that the flight is overbooked.

The repurchase mechanism

With an overbooked flight, at least one of the tickets must be repurchased by the airline. What’s interesting is that it will often be optimal for the airline to instead repurchase both tickets and leave the seat empty. To see why, note, from Eq. (3), that the airline’s repurchase mechanism is chosen to maximize

$$\Pi^2 + V^2(\theta^*)$$.

Thus, the airline is maximizing a hybrid welfare function which is the sum of producer surplus $\Pi^2$, conditioned on the average ticket holder, whose signal is either $\theta$ or $\bar{\theta}$, plus consumer surplus $V^2(\theta^*)$ conditioned on the passenger being the marginal ticket holder $\theta^*$, in this case $\theta$. Intuitively, repurchasing tickets on an overbooked flight is a transfer to ticket holders and potentially raises the value of holding a ticket, which in turn increases the price the airline can charge for tickets in the first period. On the other hand, such a transfer directly reduces the airline’s profits. The fact that these two terms are conditioned on different events creates a wedge which implies that these costs and benefits are not offset one-for-one.

To illustrate, consider first the effect on the expected payoff of a given passenger with signal $\theta$, i.e. $V^2(\theta)$, of a one dollar cash transfer to that passenger in the event that both passengers have low values. Conditional on having signal $\theta$, a passenger expects to receive this transfer with probability $\alpha$ given by

$$\alpha = g(v|\theta)f(\bar{\theta}|\theta \geq \theta)g(v|\theta).$$

The transfer thus increases $V^2(\theta)$ by $\alpha$. On the other hand, it reduces $\Pi^2$ by

$$\beta = [g(v|\theta)f(\bar{\theta}|\theta \geq \theta)]^2.$$ 

This follows because $\beta$ is the probability, conditional on both passengers purchasing tickets, that both have low values so that the airline transfers the dollar. Note that $\beta = \alpha f(\bar{\theta}|\theta \geq \theta)$ so that the net effect on the airline’s objective is positive. A one dollar
transfer increases $\Pi^2 + V^2(\theta^*)$ by $\alpha (1 - f(\theta | \theta \geq \theta)).$

It follows that the airline has an incentive to increase the size of the subsidy as much as possible. The size of the subsidy is constrained by incentive compatibility: a buyer with value $\pi$ must not prefer to claim a value $v$.

How seats are allocated affects the incentive constraint. For example, consider an efficient mechanism where exactly one of the passengers is seated if both report values $v$, with each passenger being seated with equal probability. In that case, the transfer $\tau$ to a given passenger cannot be larger than $\pi/2$. To see why, note that, if the passenger in question has value $\pi$ but reports value $v$, he would fly with probability $1/2$, receive the subsidy $\tau$, and obtain payoff $\tau + \pi/2$. Incentive compatibility requires that this payoff be no larger than the payoff from reporting $\pi$ truthfully, flying with probability one and receiving no subsidy: $\tau + \pi/2 \leq \pi$.

As an alternative, the airline could use an inefficient mechanism which seats no passenger when both ticketholders report low values. This enables the subsidy to be raised to $\pi$ without violating the incentive compatibility of a high-value passenger. We can calculate the net effect on the airline’s objective as follows. Increasing the subsidy from $\pi/2$ to $\pi$ decreases $\Pi^2$ by $\beta \pi/2$ but it increases $V^2(\theta^*)$ by

$$\alpha \left[ \pi/2 - v/2 \right].$$

The latter reflects that $\alpha$ is the probability, in the eyes of the marginal ticket holder, that the subsidy will be triggered, and in this event his subsidy increases by $\pi/2$ but his utility from flying decreases by $v/2$ (in the efficient mechanism he would have had an equal chance of retaining his seat). Thus, the distorted mechanism with a high subsidy is preferred by the airline whenever

$$(\alpha - \beta) \pi/2 - \alpha v/2 > 0$$

which, since $\beta = \alpha f(\theta | \theta \geq \theta)$, is equivalent to

$$(1 - f(\theta | \theta \geq \theta)) \pi > \pi.$$

If this holds, then it is profitable to leave the seat empty and increase the transfer to the low-value passenger.
The Spot Mechanism

Now consider the case in which no tickets were sold in the first period. In this case the airline may sell the seat in a spot auction to passengers arriving in the second period. According to Eq. (3) it will use a mechanism which maximizes

\[ f(\emptyset)\Pi^0 - [1 - f(\emptyset)]V(\theta). \] (7)

Notice that the airline is trading off increased profits from the spot mechanism against reduced welfare of a passenger with the marginal signal. The key observation is that, in equilibrium, a passenger with the marginal signal \( \theta \) will purchase a ticket and therefore will not be present in the spot market. Thus the welfare \( V^0(\theta) \) represents the hypothetical value that she would realize if she were to deviate and wait to purchase a ticket in the spot market. The airline has the incentive to distort the terms of trade in the spot market in order to make it unattractive to the marginal ticket holder.

To build intuition for the effects of this new incentive, it helps to first recall the tradeoffs present in a conventional spot auction not preceded by an initial round of ticket sales. The seller must decide whether to sell the seat in the event that the two potential buyers in the spot market both have willingness to pay \( v \). If the airline sells the seat to, say, passenger 1, this affects profits \( \Pi^0 \) in two ways. First, the direct increase in profits is the additional revenue \( v \) from the sale. The countervailing effect is the loss in revenue from the infra-marginal scenario when passenger 1 has the high value \( \bar{v} \) and passenger 2 has value \( v \). The price in that scenario would have to be reduced from \( \bar{v} \) to \( v \) in order to maintain incentive compatibility for type \( v \). Thus, conditional on the event that buyer 2 has value \( v \), the effect on profits results from the usual tradeoff between these two effects and is equal to

\[ g(v)v - g(\bar{v})(\bar{v} - v), \]

where \( g(\bar{v}) \) and \( g(v) \) represent the unconditional distribution of valuations since in the present scenario all buyers participate in the spot auction.

Thus, an airline that does not use a sequential mechanism would seat the low value passenger if and only if

\[ \frac{v}{\bar{v} - v} - \frac{g(\bar{v})}{g(v)} \geq 0. \] (8)

With advance ticket sales the criterion changes in two ways. First, the distribution of passenger values of the unticketed passengers at date 1 is \( g(\cdot|\emptyset) \), since in our mechanism types \( \theta \) and \( \bar{\theta} \) purchase tickets. Second, the airline now internalizes the effect on \( V^0(\theta) \). By selling to the low-value passenger and thereby conceding surplus \( \bar{v} - v \) to the high-value passenger, the airline makes the option of waiting for the spot market more attractive to
early-arriving passengers. In particular, the passenger with the marginal signal $\theta$ attaches probability $g(\bar{v}|\theta)$ to the event that waiting would earn him these rents. Thus, the sale to $\bar{v}$ would increase $V^0(\theta)$ by 

$$g(\bar{v}|\theta)(\bar{v} - \bar{v})$$

and thus the overall effect on the airline’s objective in the spot mechanism is

$$f(\emptyset) [g(\bar{v}|\emptyset)\bar{v} - g(\bar{v}|\emptyset)(\bar{v} - \bar{v})] - [1 - f(\emptyset)] g(\bar{v}|\emptyset)(\bar{v} - \bar{v}).$$

After re-arranging, we see that the airline seats the low-value passenger in the spot mechanism if and only if 

$$\frac{\bar{v}}{\bar{v} - \bar{v}} - \frac{g(\bar{v}|\emptyset)}{g(\bar{v}|\emptyset)} - \frac{1 - f(\emptyset)}{f(\emptyset)} \frac{g(\bar{v}|\emptyset)}{g(\bar{v}|\emptyset)} \geq 0$$

Thus, by comparison to a standard spot auction there are two effects that operate in potentially opposing directions. First, if the relative probability of high values among unticketed passengers is smaller than among ticketed passengers, then the airline has a reduced incentive to distort. This is captured by a comparison of the second terms above and in Eq. (8). On the other hand, the incentive to distort is magnified in order to increase the incentive to book early as captured by the third term above. Note that this last effect is decreasing in the probability of late-arriving passengers: when the probability of late arrival is high, information rents expected by the buyer with marginal signal $(\theta)$ are less important relative to the efficiency losses experienced by the low-value unticketed passenger.

**The Reallocation Mechanism**

Finally, consider the case in which a single ticket is purchased in the first period. According to Eq. (3), the reallocation mechanism should maximize

$$\Pi_1 + \Pi_1 + V_1(\theta^*) - \frac{1 - F(\theta^*)}{F(\theta^*)} V_1(\theta^*).$$

Let us examine the airline’s incentive to reallocate the seat from the original ticket holder to the new arrival when both passengers have low values. For the same reasons as in the re-purchase mechanism, the airline will want to buy back the ticket at the highest possible price consistent with incentive compatibility, i.e. at a price of $\bar{v}$. Thus,

$$g(\bar{v}|\emptyset) \left[ -\bar{v}g(\bar{v}|\theta \geq \theta) + (\bar{v} - \bar{v})g(\bar{v}|\theta) \right]$$

measures the effect of the repurchase on $\Pi_1 + V_1(\theta^*)$: In the event that the newcomer has value $\bar{v}$, the airline will pay $\bar{v}$ to a low-valued ticketed passenger, and will thereby increase
the rents of the low-valued ticketed passenger by \((\overline{v} - v)\). The effect on \(\Pi^1 - \frac{1 - F(\theta^*)}{F(\theta^*)} V^1(\theta^*)\) is measured by

\[
g(v|\theta \geq \overline{\theta}) \left[ v g(v|\emptyset) - g(v|\emptyset)(\overline{v} - v) - \frac{1 - f(\emptyset)}{f(\emptyset)} g(\overline{v}|\theta)(\overline{v} - v) \right], \tag{10}
\]

since in the event that the ticketholder has value \(v\) the airline has additional revenue \(v\) from a low-valued new arrival, loses revenue \((\overline{v} - v)\) from the infra-marginal sale, and raises the rent expected by \(\theta\) from foregoing the early purchase of a ticket. Thus the effect is proportional to the effect of seating a passenger in the spot mechanism, as discussed above.

The value to the airline of transferring the ticket from the current ticket holder to the newcomer is thus measured by the sum of the cost, as measured in Eq. (9) and the benefit as measured in Eq. (10). The condition in Eq. (4) guarantees that the overall effect on the airline’s profit is negative, so that the airline optimally chooses not to seat the newcomer. This does not, however, imply that the ticketed passenger will always be seated. If the virtual utility in Eq. (9) is negative, then the airline prefers to repurchase the ticket and leave the seat empty. Indeed, noting that this is equal to

\[
- \frac{g(v|\emptyset) g(v|\theta \geq \overline{\theta}) (f(\emptyset) + f(\overline{\theta}))}{f(\emptyset)} \{v - (1 - f(\emptyset|\theta \geq \overline{\theta})) \overline{v}\}, \tag{11}
\]

we see that the virtual utility replicates the tradeoffs calculated explicitly for the case with \(s = 2\) (i.e., the repurchase mechanism): no seat is filled with a low-value passenger if and only if the expression in Eq. (11) is positive.

**First Period Ticket Sales**

Having observed the optimality of selling tickets to both \(\theta\) and \(\overline{\theta}\), it is of interest to understand when this mechanism strictly improves on what is achievable by contracting only in the spot market at date 1. Rather than determining the precise parameter values, we instead illustrate some general advantages of early ticket sales and show some simple sufficient conditions that they imply.

Consider first the case in which the optimal spot auction is efficient, i.e. always sells the seat to the passenger with the highest value (breaking ties at random). In such a mechanism, an early-arriving passenger with signal \(\theta\) earns rents in expectation. This is because there is a positive probability that his realized value will be \(\overline{v}\) and the other passenger will have value \(v\). In that case the auction price will be \(\overline{v}\) and he will earn rents equal to \(\overline{v} - v\). Consider now a two-stage mechanism in which

1. Tickets are sold at a price equal to \(\theta\)'s expected rents from the spot auction.
2. Unticketed passengers are excluded.

3. Otherwise the allocation and transfer rules are the same as the spot auction.

4. In particular the value of an unticketed passenger is still used to determine the allocation and price paid by a ticketed passenger.

In this mechanism, buying a ticket is equivalent to buying admission to the spot auction. The ticket price extracts the expected rents of $\theta$, and reduces the rents of $\overline{\theta}$ by an equal amount, but leaves the welfare from trade with ticketed passengers unchanged. These effects raise profits relative to the spot auction. Any losses come from exclusion of unticketed passengers. Thus, if the probability of $\emptyset$ is sufficiently low, then this two-stage mechanism improves upon the spot auction.

Exclusion of unticketed passengers ensures that an early arriver would obtain zero utility if he were to refuse to buy a ticket. This increases the additional surplus of the marginal passenger (i.e., the passenger with signal $\theta$) from holding a ticket and this surplus will be extracted through the ticket price. Total exclusion of unticketed passengers is not necessary to achieve this. It is enough that the airline refuses to seat a low-value passenger unless he is holding a ticket. This guarantees that high-value unticketed passengers earn no information rents and again the expected utility of an unticketed passenger is zero.

Alternatively, if the optimal spot auction excludes low-value buyers, then a mechanism with ticket sales like that described above can improve profits by improving efficiency. In particular, while ticketed passengers with low values are seated under this mechanism, it still gives zero rents to a passenger with signal $\theta$. On the other hand, such a mechanism gives information rents to a passenger with signal $\overline{\theta}$. Thus, a sufficient condition for ticket sales to yield greater profits is that $\overline{\theta}$ has sufficiently low probability.\(^{25}\)

### 4 General Analysis

#### General properties of the optimal mechanism

Before describing how we approach the problem of characterizing the optimal allocation rule, we propose a system of date-1 transfers which maximizes profit for the airline given any implementable allocation rule $q$. By the envelope theorem, the date-1 mechanism defined above must satisfy, for any $s$, any passenger $i$, and any $v$,

\(^{25}\)Note that optimal profits can always be achieved by making ticket sales at date 0. The question is whether the sale of tickets allows the airline to strictly improve over a spot auction at date 1. For this example, ticket sales strictly improve profits if and only if the airline prefers to seat low-value ticket holders (i.e., if the reverse of the inequality (Eq. (6)) holds).
\[ U^s_i(v) = U^s_i(v, v_i) - \int_{v_i}^{v} q^s_i(y, v_i) dy = U^s_i(v, v_i) + \int_{v_i}^{v} q^s_i(y, v_i) dy. \] (12)

Note that the share of the surplus a passenger \( i \) expects to obtain at date one, for each \( s \) and \( v_i \), is determined up to a constant by his value \( v_i \) and the allocation rule \( q \). For ticketed passengers, adjusting this constant does not affect expected payoffs at date zero provided the ticket price is correspondingly adjusted according to Eq. (1). It is therefore without loss of optimality to focus on date-1 mechanisms such that, for all \( s \), all ticketed passengers \( j \), and all \( v_{-j} \), \( U^s_j(\overline{v}, v_{-j}) = \overline{v} \). On the other hand, adjusting period-1 payoffs for unticketed passengers by a constant does affect ex-ante payoffs. In particular, if the payoff earned by the passenger with the minimum value \( U^s_k(v, v_{-k}) \) is greater than zero, the airline can profit by increasing transfers by unticketed passengers by an appropriate constant. This not only reduces the rent left to unticketed passengers but also allows the airline to increase ticket prices according to Eq. (1). We can thus focus on mechanisms satisfying \( U^s_k(v, v_{-k}) = 0 \) for all \( s \), all unticketed passengers \( k \) and all \( v_{-k} \). The corresponding transfers are given by

\[
t^s_j(v_j, v_{-j}) = v_j q^s_j(v_j, v_{-j}) - \overline{v} + \int_{v_j}^{v} q^s_j(y, v_{-j}) dy, \\
t^s_k(v_k, v_{-k}) = v_k q^s_k(v_k, v_{-k}) - \int_{v_k}^{v} q^s_k(y, v_{-k}) dy.
\] (13)

The airline’s profit now depends only on the allocation rule. Note also from Eq. (12) that adjusting the allocation to ensure passengers with values \( \overline{v} \) fly wherever possible (i.e., setting \( q^s_i(\overline{v}, v_{-i}) = 1 \) wherever possible and adjusting transfers accordingly) does not affect passenger payoffs. Moreover, since a buyer \( i \)’s allocation remains monotonic in his valuation, the allocation remains implementable. In other words, it is possible and costless to provide the passenger with the option to fly by reporting \( \overline{v} \) whenever the number of passengers reporting \( \overline{v} \) is less than the available seats \( m \). This observation, together with Eq. (13), implies the following result.

**Proposition 1.** It is optimal to structure the mechanism so that

1. A ticket is an option to fly and passengers are never bumped involuntarily.\(^{27}\)

2. Ticket holders who are seated make (and receive) no payments.

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\(^{26}\)This monotonicity is both necessary and sufficient for the ex-post implementability of the allocations by the date-1 mechanism.

\(^{27}\)Note that passengers will not be bumped involuntarily even in case more than \( m \) passengers have value \( \overline{v} \). In this case, all of these buyers will be indifferent between keeping their seat and instead taking the compensation \( \overline{v} \) on offer.
3. Any ticket holder who does not fly receives a payment at least his value in compensation.

Part 1 of the proposition states that, at least in the above framework, airlines cannot profit from using involuntary bumping. In particular, there is no loss to airlines in allowing those with the greatest value of flying to do so. Indeed, the arrangement can be optimally structured so that ticketed passengers do not pay anything to keep their seat (Part 2 of the proposition) but are compensated for giving up a seat. Under this arrangement, enticing passengers with high values to give up their seats (as required if the flight is overbooked and the passengers with tickets have high values), while still seating those with the very high valuations, requires a high compensation to the unseated ticketed passengers. While the compensation provided by the airline in such cases may be large, the surplus provided to ticketed passengers can be recouped through the date-zero ticket price.

Note that the optimality of a mechanism without “involuntary bumping” follows from the use of sufficiently sophisticated auction mechanisms for resolving which passengers will fly. For instance, if the flight is overbooked by passengers with high values for being seated, the level of refund offered responds to their reported values and is correspondingly high. A possible disadvantage of such mechanisms in practice is that the available refund as well as the price of a seat to unticketed passengers is completely determined only at date 1 after passengers report their values. Airlines may favor simpler mechanisms where prices are specified before passengers have learned their final values for flying. If these mechanisms cannot respond to excess demand (e.g. when more ticket-holders than there are seats realize high values for flying), then involuntary bumping would be necessary for reconciling demand with the number of seats.\textsuperscript{28}

**Optimal allocations**

We now derive the optimal allocation. From Eq. (3), for each \( s \), the airline’s profits are proportional to

\[
s \left( \Pi - n \frac{1}{F(\theta^*)} V^s(\theta^*) \right) + \left( \Pi - n \right) F(\theta^*) V^s(\theta^*) .
\]  

(14)

From Eq. (12), the revenues earned from a ticketed passenger \( j \) and non-ticketed...
passenger \( k \) through the \( s \)-mechanism are

\[
\Pi^s = \int_v t^*_j(v) dG(v|S^*) = \int_v [v_j q^*_j(v) - U^*_j(v)] dG(v|S^*)
\]

and

\[
\Pi^s = \int_v [v_k q^*_k(v) - U^*_k(v)] dG(v|S^*)
\]

and the welfare of the passenger with the marginal signal is

\[
\nabla^s(\theta^*) = \int_{v_j} \int_{v_{-j}} U^*_j(v) dG(v_{-j}|S^*_{-j}) dG(v_j|\theta^*)
\]

(15)

for a ticketed passenger \( j \) and

\[
\nabla^s(\theta^*) = \int_{v_k} \int_{v_{-k}} U^*_k(v) dG(v_{-k}|S^*_{-k}) dG(v_k|\theta^*)
\]

(16)

for a passenger \( k \) without a ticket.

Focusing now on the expressions for ticketed passengers, using Eq. (12) and integration by parts yields

\[
\Pi^s = -\int_{v_{-j}} U^*_j(\bar{v}, v_{-j}) dG(v_{-j}|S^*_{-j})
\]

\[
+ \int_v q^*_j(v) v_j dG(v|S^*) + \int_{v_{-j}} \left[ \int_{v_j} G(v_j|\tilde{\theta}_j \geq \theta^*) q^*_j(v) dv_j \right] dG(v_{-j}|S^*_{-j})
\]

and

\[
\nabla^s(\theta^*) = \int_{v_{-j}} U^*_j(\bar{v}, v_{-j}) dG(v_{-j}|S^*_{-j}) - \int_{v_{-j}} \left[ \int_{v_j} G(v_j|\theta^*) q^*_j(v) dv_j \right] dG(v_{-j}|S^*_{-j}).
\]

Putting these together and collecting terms we obtain an expression for the virtual surplus of ticketed passengers

\[
s(\Pi^s + \nabla^s(\theta^*)) = E_{\bar{v}} \left[ \sum_{j=1}^s q^*_j(\bar{v}) \nabla S(\bar{v}_j) \right] |S^*],
\]

where\(^{29}\)

\[
\nabla S(v_j) = v_j - \frac{G(v_j|\theta^*) - G(v_j|\tilde{\theta}_j \geq \theta^*)}{g(v_j|\tilde{\theta}_j \geq \theta^*)}.
\]

\(^{29}\)Remember that \( G(v_j|\tilde{\theta}_j \geq \theta^*) \) is the distribution of passenger \( j \)'s value, conditional event of having signal \( \tilde{\theta}_j \geq \theta^* \). That is, distribution of values, conditional on having purchased a ticket.

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As anticipated above, all terms involving $U^*_j(\bar{v}, \bar{v}_{-j})$ are canceled: these constants are free variables in the airline’s maximization.\footnote{As described above, this is intuitive because any constant added to the ticket holder’s ex post utility at date 1 can be recovered via an equal increase in the ticket price at date 0.}

Turning now to the terms in Eq. (14) involving unticketed passengers, and using that unticketed passengers with the lowest values earn zero surplus, we have
\[
(n - s) \left( \Pi^* - \frac{1 - F(\theta^*)}{F(\theta^*)} V^*(\theta^*) \right) = \mathbb{E}_q \left[ \sum_{k=s+1}^n q^*_k(\bar{v}) \overline{V}(\bar{v}_k) \right] S^*,
\]

where\footnote{Remember that $G(v_k|\bar{\theta}_k < \theta^*)$ is the distribution of values $v_k$, conditional on event $\bar{\theta}_k < \theta^*$. That is, distribution of values, conditional on not having a ticket (either because signal was low or consumer $k$ was not present in date 0.)}
\[
\overline{V}(v_k) = v_k - \frac{1 - G(v_k|\bar{\theta}_k < \theta^*) + \frac{1 - F(\theta^*)}{F(\theta^*)} (1 - G(v_k|\theta^*))}{g(v_k|\theta_k < \theta^*)}.
\]

To summarize the above, we have derived a convenient expression for the airline’s profit in an optimal mechanism implementing an allocation rule $q$ which satisfies the requirements set out in Section 3.

**Lemma 3.** Let $s$ be a given number of tickets sold and consider the date-1 mechanism implementing the allocation $q^*(\cdot)$. Suppose that the unticketed passenger with the lowest value earns zero surplus in this mechanism. Then the expression in Eq. (14) is equal to
\[
\mathbb{E} \left[ \sum_{j=1}^s q^*_j(\bar{v}) \overline{V}(\bar{v}_j) + \sum_{k=s+1}^n q^*_k(\bar{v}) \overline{V}(\bar{v}_k) \right] S^*. \tag{17}
\]

The transformed expected revenue of Eq. (17) is a familiar expected virtual surplus. As usual, the first pass at a solution is to consider the allocation rule which, at every realized valuation profile, allocates seats to the $m$ passengers with the highest (non-negative) virtual surpluses. One then verifies that the proposed allocation indeed gives precedence to ticketed passengers over unticketed passengers in the sense described in Section 3, and that the ticket price defined by Eq. (1) and transfers defined by Eq. (13) implement the allocation. The following conditions on the virtual surpluses are sufficient to guarantee this.

**Proposition 2.** Fix $\theta^* \in [0, 1]$. Suppose that (i) $\max \{\overline{V}(v_j), 0\}$ and $\max \{\overline{V}(v_k), 0\}$ are non-decreasing functions, and (ii) $\overline{V}(v_i) \geq \overline{V}(v_i)$ for all $v_i \in \text{Supp} \left[ G(\cdot|\bar{\theta} \geq \theta^*) \right] \cap \text{Supp} \left[ G(\cdot|\bar{\theta} < \theta^*) \right]$, a nonempty set.\footnote{The requirement that $\text{Supp} \left[ G(\cdot|\bar{\theta} \geq \theta^*) \right] \cap \text{Supp} \left[ G(\cdot|\bar{\theta} < \theta^*) \right]$ be nonempty is a minor technical} The allocation which maximizes virtual surplus
(conditional on the marginal signal $\theta^*$) is implementable and maximizes the airline’s profit.

Condition (i) ensures that an extended allocation can be found such that each passenger’s probability of being seated is non-decreasing in his own report (whether or not that report is made in equilibrium). It thus guarantees the existence of an ex-post incentive-compatible period-1 mechanism which implements the allocation. Condition (ii) ensures that an allocation $q$ can be found that gives precedence to ticketed passengers over unticketed passengers in the sense described in Section 3. Given that the ticket price satisfies Eq. (1), this guarantees passengers purchase tickets if and only if their signal exceeds $\theta^*$.

Note that, given the conditions of Proposition 2, the optimal pricing mechanism has the property that, if there are two passengers with the same value and the one without a ticket is seated, then the passenger with a ticket must be seated as well. This is a different sense in which ticketed passengers are given precedence relative to unticketed passengers (we make the same point in footnote 15 above).

The above analysis takes the marginal signal $\theta^*$ as given, and then finds the optimal allocation conditional on this cut-off. The final step in deriving the profit-maximizing pricing mechanism is to deduce the optimal choice of $\theta^*$. In the following example we are able to find an analytical expression for it.

Example 1. Suppose that, for each passenger $i$, $\tilde{\theta}_i$ and $\tilde{\varepsilon}_i$ are independently and uniformly distributed on $[0, 1]$. Suppose that passengers’ values are determined by $v_i = \theta_i + a\varepsilon_i$, where $a \in (0, 1/4)$. In the absence of capacity constraints (i.e., if $m \geq n$), the profit-maximizing choice of the marginal signal is $\theta^* = \frac{1}{2} - \frac{a}{4}$.

More generally, selling tickets (i.e., setting $\theta^* < 1$) improves profits if the airline profits from choosing a more favorable allocation for ticketed as opposed to unticketed passengers. We therefore have the following corollary to Proposition 2.

Corollary 1. Suppose that there exists $\theta^* \in [0, 1)$ such that (i) max $\{\overline{V}(v_j), 0\}$ and max $\{\underline{V}(v_k), 0\}$ are non-decreasing functions, and (ii) $\overline{V}(v_i) \geq \underline{V}(v_i)$ for all $v_i \in \text{Supp} \left[ G \left( \cdot | \tilde{\theta} \geq \theta^* \right) \right] \cap \text{Supp} \left[ G \left( \cdot | \tilde{\theta} < \theta^* \right) \right]$, a nonempty set, with a strict inequality for an interval of positive length. The airline strictly prefers a mechanism with ticket sales at date 0 as opposed to one without ticket sales.

The usefulness of the ticket sales as an additional screening device depends on the accuracy of the signals buyers receive about their valuations. In the extreme case where requirement, which is satisfied in most cases of interest. For instance, it is enough to suppose that type $\varnothing$ has positive probability and that $G (\cdot | \varnothing)$ has full support on $[\underline{v}, \overline{v}]$.

33The conditions ensure that there are some values for which, under the optimal allocation given $\theta^*$, holding a ticket matters for whether a passenger is seated. While we find these conditions straightforward to give in light of Proposition 2, we expect one can also find weaker conditions to guarantee the profitability of offering tickets in advance of the flight.
the buyers learn their valuations with certainty at date 0, selling tickets does not increase airline profits (and hence overbooking is not required to maximize profits).

**Example 2.** When signals are fully informative, i.e. passengers arriving at date zero know their values with certainty at that date, a spot auction at date 1 is an optimal mechanism.

**Understanding the trade-offs**

**Ticketed Passengers**

We now consider the allocations described in Proposition 2 and how these reflect the airline’s incentives. The virtual surplus of ticketholders is given by

\[
\text{VS}(v_j) = v_j - G(v_j | \theta^*) - G(v_j | \tilde{\theta}_j \geq \theta^*) - g(v_j | \tilde{\theta}_j \geq \theta^*)
\]

To interpret this, first consider that the decision not to seat a ticketed passenger \(j\), i.e. by setting \(q^*_s(v) = 0\) for some \(s\) and realized values \(v\), is an advance commitment to a ticket purchaser \(j\) that his ticket will be repurchased in the event that \(s\) passengers hold tickets and the profile of all passengers’ realized valuations is \(v\). Such a commitment impacts the airline’s profits via two effects. First, it raises the payment to a ticketed passenger announcing value \(v_j\) and thus, by incentive compatibility, requires the airline to raise by an equal amount the utility of all types lower than \(v_j\). This directly reduces the airline’s profits within the date 1 mechanism by \(G(v_j | \tilde{\theta}_j \geq \theta^*)\) and correspondingly lowers the airline’s willingness to repurchase the ticket from passenger \(j\). Note that \(G(v_j | \tilde{\theta}_j \geq \theta^*)\) measures the probability that \(j\)’s value will fall below \(v_j\) conditional on \(j\)’s purchase of a ticket at date 0.

The second effect operates indirectly via revenues from date 0 ticket sales. Indeed, some of the additional utility provided by the airline through date 1 repurchases can be recouped via increased ticket prices. At the time of ticket purchase, the marginal type \(\theta^*\) assesses a probability \(G(v_j | \theta^*)\) that his value will fall below \(v_j\) and he will benefit from the increased utility resulting from the airline’s commitment to repurchase a ticket from type \(v_j\). That leads to an increased willingness to pay for a ticket that can be extracted dollar-for-dollar by increasing the price of a ticket. This indirect effect raises the airlines profits by \(G(v_j | \theta^*)\) where, crucially, this measures the probability that \(j\)’s value will fall below \(v_j\) conditional on \(j\)’s purchase of a ticket at date 0.

Putting all of this together, the airline chooses to seat a passenger based on the extent to which the surplus created \(v_j\) exceeds a measure that accounts for the net effect on the airline’s profits. That measure is proportional to the difference \(G(v_j | \theta^*) - G(v_j | \tilde{\theta}_j \geq \theta^*)\).
in the repurchase probability assessed by the marginal and average ticket purchasers (θ* and θ_j ≥ θ* respectively.)

In light of this wedge between marginal and average ticketholders, overbooking and repurchasing can be seen as an instrument to refine the screening of passengers by willingness to pay at date 0. For illustrative purposes, consider an airline that practices no overbooking and seats all ticketed passengers. One way the airline could seek to increase profits is to raise the ticket price, thus reducing the surplus earned by infra-marginal ticket holders. However, doing so affects the decision of the marginal ticket holders to purchase tickets, so it comes at the cost of reduced ticket sales.

Overbooking enables the airline to capture consumer surplus without sacrificing ticket sales. The airline raises ticket prices and effectively strikes a deal with the marginal type θ* that the price increase will be returned in expectation through repurchases in the event that his valuation turns out to be low. Since this is calculated to be a one-for-one intertemporal transfer for the marginal type θ*, it is less favorable for passengers with signals above θ* because they assess a strictly lower probability of repurchase. Thus, the higher price coupled with repurchases maintains the indifference of the marginal type and strictly lowers the consumer surplus of infra-marginal types.

**Unticketed Passengers** Consider now the virtual surplus of an unticketed passenger.

\[
\text{VS}(v_k) = v_k - \frac{1 - G(v_k|\theta_k < \theta^*) + \frac{1-F(\theta^*)}{F(\theta^*)} (1 - G(v_k|\theta^*))}{g(v_k|\theta_k < \theta^*)}
\]

Just as with ticketed passengers, the incentive to seat an unticketed passenger mixes the direct effect on revenues in the s-mechanism with an indirect effect on revenues from ticket sales. Indeed, the virtual surplus can be seen as the sum of two terms, the first of which,

\[
v_k - \frac{1 - G(v_k|\theta_k < \theta^*)}{g(v_k|\theta_k < \theta^*)}
\]

is the familiar expression for the marginal revenue from selling to buyer k in a standard monopoly problem. It has the one noteworthy difference that the expressions are conditioned on passenger k having a date 0 signal below the threshold θ*. The remaining term in VS is a correction which accounts for the effect on first period ticket sales from a date 1 decision to seat an unticketed passenger. A mechanism which allocates space to unticketed passengers reduces the value of holding a ticket and hence revenue from ticket sales. This term conditions on θ* because it is the marginal ticketholder’s willingness to pay that determines the ticket price.
Implementation via a double auction

Assuming Proposition 2 applies, it is straightforward to specify a “handicap” double auction which implements the optimal pricing mechanism. The transfers that are made in the auction we propose are identical to those given by Eq. (13).

Some new notation will be convenient for formalizing the rules of the auction. For any ticket-holder value $v_j$, define the matching value $v_k(v_j)$ for an unticketed passenger to be the highest value satisfying

$$\nabla S(v_j) = \nabla S(v_k(v_j)).$$

Note that $v_k(v_j)$ is increasing in $v_j$. Conversely, define $v_j(v_k)$ as the matching value of the ticket holder. The rules of the double auction are as follows. Each passenger submits a bid and the airline announces the reserve price $R$ defined to be the highest value satisfying $\nabla S(R) = 0$. Passengers are ranked in descending order of bids and the $q$ highest bidding passengers are allocated a seat provided their bid exceeds the reserve $R$.

Payments are determined as follows. Let $b^q$ and $b^{q+1}$ denote the $q$ and $q + 1$st highest bids. In case the number of bids exceeding $R$ is smaller than either $q$ or $q + 1$, then $b^q$ and/or $b^{q+1}$ are set equal to $R$. Any ticketed passenger who is not seated receives compensation equal to $b^q$. Any unticketed passenger who wins a seat is charged $v_k(b^{q+1})$. The transfers are zero for all ticketed passengers who fly and unticketed passengers who do not.

**Proposition 3.** Suppose that the conditions of Proposition 2 hold. Then there exists a dominant strategy equilibrium of the double auction which implements the optimal pricing mechanism.

Note that the spread between the price paid by unticketed passengers for seats and the compensation to the unseated ticket holders need not be positive. Although seats are transferred from ticketed passengers to unticketed passengers only if the latter have sufficiently higher values than the former, it does not mean that the net transfers are always positive. To see this, consider an example with one seat, one ticketed passenger and one unticketed passenger. Suppose ticketed passenger has close to the lower bound $v$ and unticketed passenger value close to the upper bound $\overline{v}$. Then the ticketed passenger may receive compensation close to $\overline{v}$, whereas the unticketed passenger may pay a significantly smaller amount for the seat.\(^{34}\) However, the negative ex-post net revenue occurs only if the difference between $b^q$ and $b^{q+1}$ is sufficiently large and these bids are made by an unticketed and a ticketed passenger respectively. The probability of this event is very low, when the number of passengers is higher than two.

\(^{34}\)The argument is analogous to Myerson and Satterthwaite (1983).
5 Conclusions

This paper proposed an incentive-based rationale for overbooking and shed light on a novel interplay between screening ex-ante via ticket prices and screening ex-post by auctions. Compensation offered to ticketed passengers in return for their seats helps improve demand for tickets, allowing the airline to raise ticket prices. At the same time, passengers who strongly believe they will have a high value for flying do not expect to benefit from such compensation. These passengers, who are infra-marginal when it comes to buying tickets, pay the higher ticket prices but expect to benefit little from compensation, so they expect less surplus.

Repurchased tickets may or may not be sold to unticketed passengers with high values for flying. Selling seats to unticketed passengers can improve efficiency, but it also improves the outside option of passengers who may consider purchasing tickets in advance of the flight, lowering the demand for tickets.

To highlight our main ideas, we have abstracted from several relevant features of airline markets. The restriction to a single fare class is a simplification, although in the Appendix we show how our analysis can be extended to multiple classes. While we focused on a monopolist airline, overbooking is likely to have implications for competition. A buyer who has purchased a ticket from one firm presumably finds it very costly to also purchase a ticket from a competitor, even if there is a positive probability that she will be bumped. We assumed that passengers are forward-looking and strategic and abstracted from possible behavioral biases. For instance, the extent to which passengers internalize possible compensation when considering ticket purchases clearly has important implications for our theory.
References


A Proofs of results

This Appendix collects proofs not given in the main text.

Proof of Lemma 1

Let \( \Theta_T \) be the set of signals for which passengers purchase tickets, which may be any measurable subset of \([0,1]\). We need to show that there exists \( \theta^* \in [0,1] \) such that \( \Theta_T = [\theta^*, 1] \).

First note that the stochastic process described by \( F \) and \( G \) necessarily admits an “independent-shock” representation as follows (see Eso and Szentes (2007) and Pavan et al. (2013)). Define, for each \((\theta, \varepsilon) \in [0,1]^2\), \( z(\theta, \varepsilon) = G^{-1}(\varepsilon|\theta) \). That is, for each \((\theta, \varepsilon) \in [0,1]^2\), \( z(\theta, \varepsilon) \) is the unique value in the support of \( G(\cdot|\theta) \) satisfying \( G(z(\theta, \varepsilon)|\theta) = \varepsilon \). Suppose that \( \tilde{\varepsilon} \) is uniformly distributed on \([0,1]\). Then, for each \( \theta \in [0,1] \), \( z(\theta, \tilde{\varepsilon}) \) is distributed according to \( G(\cdot|\theta) \).

Without loss of generality, we focus on direct mechanisms at date 1 which induce truthtelling irrespective of the decision to buy a ticket at date 1. Let \( q^s(v_j) \) be the probability a ticketed passenger \( j \) is seated conditional on \( s \) tickets being sold, and his reported value being \( v_j \); and let \( q^s(v_k) \) be the probability of being seated for an unticketed passenger \( k \). Let \( V^s(\theta_j) \) be the expected payoff (gross of the ticket price) to a ticketed passenger conditional on \( s \) tickets being purchased and let \( V^s(\theta_j) \) be the expected payoff to an unticketed passenger. Let \( \tilde{r} \) be distributed according to a binomial distribution with parameters \((n-1, 1 - F(\Theta_T))\), where \( F(\Theta_T) \) is the probability that any other passenger purchases a ticket.

Applying Theorem 1 of Pavan et al. (2013), we have that the expected payoff for a passenger who purchases a ticket satisfies, for any signals \( \theta', \theta'' \in [0,1] \) with \( \theta' < \theta'' \),

\[
E_{\tilde{r}} \left[ V^{\tilde{r}+1}(\theta'') \right] = E_{\tilde{r}} \left[ V^{\tilde{r}+1}(\theta') \right] + \int_{\theta'}^{\theta''} E_{(\tilde{r}, \tilde{\varepsilon})} \left[ \frac{\partial z(y, \tilde{\varepsilon})}{\partial \theta} q^{\tilde{r}+1}(z(y, \tilde{\varepsilon})) \right] dy. \tag{18}
\]

The expected payoff for a passenger who does not purchase a ticket satisfies

\[
E_{\tilde{r}} \left[ V^{\tilde{r}}(\theta'') \right] = E_{\tilde{r}} \left[ V^{\tilde{r}}(\theta') \right] + \int_{\theta'}^{\theta''} E_{(\tilde{r}, \tilde{\varepsilon})} \left[ \frac{\partial z(y, \tilde{\varepsilon})}{\partial \theta} q^{\tilde{r}}(z(y, \tilde{\varepsilon})) \right] dy. \tag{19}
\]

That higher signals imply higher values in the sense of first-order stochastic dominance means that, for all \((\theta, \varepsilon)\), \( \frac{\partial z(\theta, \varepsilon)}{\partial \theta} \geq 0 \). Our assumption that ticket holders are favored implies that, for all \( s \in \{0, \ldots, n-1\} \) and all \( v_i \in [\underline{v}, \overline{v}] \), \( q^s(v_i) \leq \overline{q}^{s+1}(v_i) \). Hence, for
\[ \theta', \theta'' \text{ with } \theta' < \theta'', \]

\[ \int_{\theta'}^{\theta''} \mathbb{E}_{(\hat{r}, \tilde{z})} \left[ \frac{\partial z(y, \tilde{z})}{\partial \theta} q_{\hat{r}}^{\hat{r}+1}(z(y, \tilde{z})) \right] dy \geq \int_{\theta'}^{\theta''} \mathbb{E}_{(\hat{r}, \tilde{z})} \left[ \frac{\partial z(y, \tilde{z})}{\partial \theta} q_{\hat{r}}(z(y, \tilde{z})) \right] dy. \tag{20} \]

Now, suppose \( p \) is the price of the ticket and that

\[ \mathbb{E}_{\hat{r}} \left[ V_{\hat{r}}^{\hat{r}+1}(\theta') \right] - p \geq \mathbb{E}_{\hat{r}} \left[ V_{\hat{r}}^\theta(\theta') \right] \]

so that a passenger with signal \( \theta' \) does better by purchasing the ticket at price \( p \) than by not purchasing. Then Equations (18), (19), and (20) imply

\[ \mathbb{E}_{\hat{r}} \left[ V_{\hat{r}}^{\hat{r}+1}(\theta'') \right] - p \geq \mathbb{E}_{\hat{r}} \left[ V_{\hat{r}}^\theta(\theta'') \right] \]

so that \( \theta'' \) also prefer does better to purchase the ticket at price \( p \). Hence, given our assumption that the passenger purchases a ticket whenever willing, if \( \theta' \) purchases, so too does \( \theta'' \). This establishes that ticket sales have a threshold property: the fact that this set is closed – i.e. includes the threshold value \( \theta'' \) – follows from the continuity of \( \mathbb{E}_{\hat{r}} \left[ V_{\hat{r}}^{\hat{r}+1}() \right] \) and \( \mathbb{E}_{\hat{r}} \left[ V_{\hat{r}}^\theta() \right] \), which is immediate from Equations (18) and (19).

**Proof of Lemma 2**

The proof is a minor adaptation of Gershkov et al. (2012) and here we provide only a sketch. Let \( \chi_i = 1 \) if \( i \) holds a ticket (i.e., \( i \leq s \)) and \( \chi_i = 0 \) otherwise (i.e., if \( i > s \)).\(^{35}\)

We let \((\tilde{q}, \hat{t})\) be a Bayesian incentive-compatible mechanism and consider the set \( C \) of alternative allocation rules \( q \) satisfying

\[ q_{\hat{t}}^s(v) \geq 0 \quad \forall v, s, i \]

\[ \sum_i q_{\hat{t}}^s(v) \leq m \quad \forall v, s \]

\[ \mathbb{E} \left[ q_{\hat{t}}^s(v_i, \tilde{v}_{-i}) \chi_i \right] = \mathbb{E} \left[ \tilde{q}_{\hat{t}}^s(v_i, v_{-i}) \chi_i \right] \quad \forall v_i, \chi_i \tag{21} \]

\[ \mathbb{E} \left[ q_{\hat{t}}^s(v) \right] = \mathbb{E} \left[ \tilde{q}_{\hat{t}}^s(v) \right]. \tag{22} \]

Let \( q^* \) solve

\[ \min_{q \in C} \mathbb{E} \left[ ||q_{\hat{t}}^s(\tilde{v})||^2 \right]. \]

By following exactly the same steps as in Gershkov et al. (2012), one can show the

\(^{35}\)Clearly, each passenger knows whether he holds a ticket, i.e. whether he is ordered below or above \( s \), but faces a distribution over \( s \) conditional on whether he holds a ticket.
existence of \( q^* \) which satisfies, for each \( i \),

\[
q_i^{s^*} (v_{-i}) \text{ is non-decreasing for all } s \text{ and } v_{-i}.
\]

That is, the probability with which \( i \) is awarded a seat is ex-post monotone in \( v_i \). By standard arguments, this implies that \( q^* \) is implementable in dominant strategies. Eq. (21) implies \( q^* \) yields the same interim expected utilities as \( q' \), and Eq. (22) is used to show that expected revenues are also the same (again following the steps in Gershkov et al. (2012)).

\[ \square \]

## Derivation of Equation (3)

The passenger with the marginal signal is indifferent between expected value from ticket 
\[ \mathbb{E}[\hat{V}^{\hat{r}+1}(\theta^*)] - p \] and expected value from not having a ticket \( \mathbb{E}[\hat{V}^\hat{r}(\theta^*)] \), where the number of other ticket holders \( \hat{r} \) has binomial distribution with parameters \((n - 1, 1 - F(\theta^*))\). This gives

\[
p = \sum_{r=0}^{n-1} \binom{n - 1}{r} (1 - F(\theta^*))^r F(\theta^*)^{n-1-r} [\hat{V}^{\hat{r}+1}(\theta^*) - \hat{V}^\hat{r}(\theta^*)].
\]

Expected total profit is
\[
\pi = n(1 - F(\theta^*))p + \mathbb{E}[s\Pi^s + (n - s)\Pi^s],
\]
where \( s \) has binomial distribution with parameters \((n, 1 - F(\theta^*))\). Therefore

\[
\pi = n(1 - F(\theta^*))p + \sum_{s=0}^{n} \binom{n}{s} (1 - F(\theta^*))^s F(\theta^*)^{n-s} [s\Pi^s + (n - s)\Pi^s].
\]

From Eq. (1), \( n(1 - F(\theta^*))p = n(1 - F(\theta^*))\mathbb{E}[\hat{V}^{\hat{r}+1}(\theta^*)] - n(1 - F(\theta^*))\mathbb{E}[\hat{V}^\hat{r}(\theta^*)] \). Rewriting the two parts of this expression,

\[
n(1 - F(\theta^*))\mathbb{E}[\hat{V}^{\hat{r}+1}(\theta^*)] = \sum_{r=0}^{n} \frac{n!}{r!(n - r)!} (1 - F(\theta^*))^r F(\theta^*)^{n-r} r\hat{V}^\hat{r}(\theta^*),
\]

\[
n(1 - F(\theta^*))\mathbb{E}[\hat{V}^\hat{r}(\theta^*)] = \frac{1 - F(\theta^*)}{F(\theta^*)} \sum_{r=0}^{n} \frac{n!}{r!(n - r)!} (n - r)(1 - F(\theta^*))^r F(\theta^*)^{n-r} \hat{V}^\hat{r}(\theta^*).
\]

Inserting \( n(1 - F(\theta^*))\mathbb{E}[\hat{V}^{\hat{r}+1}(\theta^*)] \) and \( n(1 - F(\theta^*))\mathbb{E}[\hat{V}^\hat{r}(\theta^*)] \) into the profit equation gives Eq. (3).
Proof that the mechanism described in Section 3 achieves the highest profit attainable by any mechanism

We consider without any restriction the direct mechanism. Here, with an abuse of notation, \(q_i(\theta_i, v_i; \theta_{-i}, v_{-i})\) specifies, for each passenger \(i\), the probability \(i\) is seated given the sequence of own reports \((\theta_i, v_i)\) and reports of the other passenger \((\theta_{-i}, v_{-i})\).

Let \(x_i(\theta_i, v_i) = E_{(\theta_{-i}, \tilde{v}_{-i})}[q_i(\theta_i, v_i; \tilde{\theta}_{-i}, \tilde{v}_{-i})]\). Let \(W(\theta)\) be the expected payoff to a passenger with signal \(\theta\) who reports optimally in the direct mechanism (and whose only information is his own type \(\theta\) and his initial prior as to the other passenger’s type), and \(W(\theta, v)\) the expected payoff when he receives signal \(\theta\) and his value is \(v\) (again, given he does not update his beliefs about the other passenger’s information).

First, note that, because the passenger with signal \(\tilde{\theta}\) can follow a strategy of reporting \(\tilde{\theta}\) and then \(\tilde{v}\), we must have

\[
W(\tilde{\theta}) \geq W(\tilde{\theta}) + x_i(\tilde{\theta}, v)(g(\tilde{v}\mid \tilde{\theta}) - g(\tilde{v}\mid \bar{\theta}))(\tilde{v} - \bar{v}).
\]

Because the passenger with signal \(\tilde{\theta}\) can follow a strategy of waiting and reporting only at date 1, and at that point reporting value \(\tilde{v}\), we have

\[
W(\tilde{\theta}) \geq W(\emptyset, \tilde{v}) + x_i(\emptyset, \tilde{v})g(\tilde{v}\mid \tilde{\theta})(\tilde{v} - \bar{v}).
\]

Because a passenger who arrives at date 1 and has value \(\bar{v}\) can report \(\tilde{v}\), we must have\(^{36}\)

\[
W(\emptyset, \bar{v}) \geq W(\emptyset, \tilde{v}) + x_i(\emptyset, \tilde{v})(\tilde{v} - \bar{v}).
\]

As a result, using that \(W(\emptyset, \tilde{v}) \geq 0\), the expected contribution of passenger \(i\) to the airline’s profit is at most \(E[VS(\tilde{\theta}, \bar{v}) x_i(\tilde{\theta}, \tilde{v})]\), where \(VS(\tilde{\theta}, \bar{v}) = \bar{v} - \frac{f(\bar{v})}{f(\tilde{v})}(\bar{v} - \tilde{v})\), \(VS(\emptyset, \bar{v}) = \bar{v} - \left(\frac{g(\bar{v}\mid \emptyset)}{g(\bar{v}\mid \emptyset) + (1 - f(\tilde{v})/f(\bar{v}))g(\bar{v}\mid \emptyset)}\right)(\bar{v} - \tilde{v})\), and \(VS(\tilde{\theta}, \bar{v}) = \bar{v}\) for all \(\theta\). The allocation proposed in the text maximizes the sum of expected virtual surpluses \(VS\) across both passengers, and thereby attains the maximum possible expected profit for the airline in an incentive-compatible mechanism.

It remains to consider the ticket price and date-1 payments which implement this allocation. These are chosen so that, on path, an unticketed low value passenger \(i\) pays \(q_i(\emptyset, \tilde{v}; \theta_{-i}, v_{-i})\bar{v}\) and a high-value passenger \(q_i(\emptyset, \bar{v}; \theta_{-i}, v_{-i})\bar{v} - q_i(\emptyset, \tilde{v}; \theta_{-i}, v_{-i})(\tilde{v} - \bar{v})\) for the possibility to be seated. Also, a ticketed passenger \(i\) receives payments at date 1 equal to \((1 - q_i(\theta_i, v_i; \theta_{-i}, v_{-i}))\tilde{v}\). In particular, a ticketed passenger receives compensation equal to \(\bar{v}\) times the probability he loses his seat, conditional on reported values. Via

\(^{36}\)We focus initially on interim incentive constraints, and then verify that the proposed optimal mechanism respects ex-post incentive constraints.
Eq. (1), this pins down the ticket price
\[ \bar{v} - x_i(\bar{\theta}, v) g(v | \bar{\theta}) (\bar{v} - v) - x_i(\emptyset, v) g(v | \emptyset) (\bar{v} - v). \]

Given this ticket price and transfers, the only incentive constraint that requires some consideration is that \( \bar{\theta} \) wishes to purchase a ticket at date 0. Note that, by construction of ticket prices and compensation, the expected payoff for \( \bar{\theta} \) by obediently purchasing a ticket and then reporting truthfully is
\[ x_i(\emptyset, v) g(v | \emptyset) (\bar{v} - v) + x_i(\bar{\theta}, v) g(v | \bar{\theta}) (\bar{v} - v). \]
The deviation is unprofitable if and only if this is at least what can be obtained by not purchasing a ticket, i.e. \( x_i(\emptyset, v) (\bar{v} - v) \). Therefore, purchasing the ticket is incentive compatible iff \( x_i(\bar{\theta}, v) \geq x_i(\emptyset, v) \). This is guaranteed by Condition (4) in the main text.

**Proof of Proposition 2**

The proof proceeds by specifying an allocation \( q \) which maximizes the virtual surplus Eq. (17) and which is implementable by a pricing mechanism which satisfies the restrictions set out in Section 3. For reported values consistent with equilibrium ticket purchasing, the allocation is determined simply by a comparison of the virtual surpluses. What must still be specified is the allocation when one of the passengers deviates by reporting a value inconsistent with his decision to purchase or not to purchase a ticket (the allocation in case two or more passengers make such reports is irrelevant because we need not consider joint deviations from equilibrium play).

- First, for any \( s \) and for any ticket holder \( j \) reporting \( v_j \) above the support of \( G(\cdot | \bar{\theta} \geq \theta^*) \), and for any \( v_{-j} \), \( q^*_j(v_j, v_{-j}) = 1 \) while \( q^*_i(v_j, v_{-j}) = 0 \) for all \( i \neq j \).
- Second, for any \( s \) and for any ticket holder \( j \) reporting \( v_j \) below the support of \( G(\cdot | \bar{\theta} \geq \theta^*) \), and for any \( v_{-j} \), the allocation is the same as for the passenger \( j \) report equal to the minimum of the support of \( G(\cdot | \bar{\theta} \geq \theta^*) \).
- Third, for any \( s \) and for any non-ticket holder \( k \) reporting \( v_k \) above the support of \( G(\cdot | \bar{\theta} < \theta^*) \), and for any \( v_{-k} \), the allocation is the same as in case \( k \) reports the maximum of the support of \( G(\cdot | \bar{\theta} < \theta^*) \).
- Finally, for any \( s \) and for any non-ticket holder \( k \) reporting \( v_k \) below the support of \( G(\cdot | \bar{\theta} < \theta^*) \), and for any \( v_{-k} \), \( q^*_j(v_k, v_{-k}) = 0 \) and \( q^*_i(v_k, v_{-k}) = 0 \) for all \( i \neq k \).
Using Condition (i) of the proposition, the allocations defined above satisfy, for any \( s \), any passenger \( i \) (whether or not he has purchased a ticket), and any \( v_i \) consistent with equilibrium ticket purchasing, \( q_i^s(\cdot, v_{-i}) \) is non-decreasing over \([\underline{v}, \overline{v}]\). This guarantees that the period-1 mechanism, defined by the above allocations and the transfers Eq. (13), is ex-post incentive compatible irrespective of whether the passenger follows the equilibrium strategy in purchasing (or not purchasing) a ticket.

Second, Condition (ii), together with the allocations defined above for out-of-equilibrium play, ensures that the allocation rule gives precedence to ticket holders. Since the price satisfying Eq. (1) keeps the passenger with signal \( \theta^* \) indifferent between purchasing and not purchasing a ticket, the same argument as in the proof of Lemma 1 implies that all passengers with signals above \( \theta^* \) are willing to purchase tickets, while passengers with signals less than \( \theta^* \) are not.

Proof of Example 1

Given the absence of capacity constraints, we can focus on the case with \( n = 1 \). For each \( \theta^* \in [0, 1) \), let \( \bar{q}(v) \) and \( \tilde{q}(v) \) indicate whether the passenger is seated under the optimal allocation when his value is \( v \), and when he is ticketed and unticketed, respectively. Following the same steps as in the derivation of the unrestricted mechanism (see below), the airline’s expected profit, conditional on \( \theta^* \), can be shown to equal

\[
E_{(\tilde{\theta}, \tilde{\epsilon})} \left[ \begin{array}{c}
1_{\tilde{\theta} \geq \theta^*} \bar{q}(\tilde{\theta} + a\tilde{\epsilon}) \left(2\tilde{\theta} + a\tilde{\epsilon} - 1\right) + (1 - 1_{\tilde{\theta} \geq \theta^*}) \tilde{q}(\tilde{\theta} + a\tilde{\epsilon}) \left(2\tilde{\theta} + a\tilde{\epsilon} - 1\right)
\end{array} \right].
\]  

(25)

The first-order necessary condition for an optimum is

\[
E_{\tilde{\theta}} \left[ -\nabla S (\theta^* + a\tilde{\epsilon}) \bar{q}(\theta^* + a\tilde{\epsilon}) + vV S (\theta^* + a\tilde{\epsilon}) vq (\theta^* + a\tilde{\epsilon}) \right] = 0.
\]

This yields \( \theta^* = \frac{1}{2} - \frac{a}{4} \). It can further be verified that the expected profit in Eq. (25) is quasi-concave in \( \theta^* \), so that \( \frac{1}{2} - \frac{a}{4} \) achieves the optimum.

Proof of Corollary 1

Let \( q^* \) be a symmetric allocation specifying, for each passenger \( i \), each \( v \), a probability of flying \( q_i^s(v) \) and maximizing the airline’s profit in a mechanism without ticket sales (i.e., where passengers contract only at date 1). Pick \( \theta^* \in [0, 1) \) satisfying the conditions in the Corollary. Consider now the pricing mechanism which is optimal conditional on (i) selling tickets to passengers iff their signals exceed \( \theta^* \), and (ii) implementing the symmetric
allocation rule $q^*$ (which depends only on values reported at date 1 and not on which passengers hold tickets).

From Eq. (18) and Eq. (19), we see that, given the passenger with the marginal signal $\theta^*$ is indifferent between purchasing and not purchasing a ticket, the passengers with all other signals in $[0, 1]$ must be indifferent as well. Hence, the optimal pricing mechanism with threshold $\theta^*$ and allocation $q^*$ and the mechanism without tickets both provide the same expected payoff to passengers. Since total welfare is also the same for each mechanism, total profit must be identical as well.

Note, however, that by Condition (ii) of the Corollary, the allocation $q^*$ fails to maximize the expression in Eq. (17). Hence, both the pricing mechanism that implements this allocation and the optimal mechanism without tickets deliver a strictly lower profit than the one achievable by selling tickets to passengers with signals above $\theta^*$ and by implementing an allocation which optimally distinguishes ticketed from unticketed passengers (as described in the proof of Proposition 2).

**Proof of Example 2**

This follows from noticing that, for any $\theta^* \in [0, 1]$, the expected virtual surplus Eq. (17) can be maximized by an allocation rule which is symmetric and independent of which passengers are ticketed. The optimal mechanism without ticket sales, where passengers contract only date 1, gives passengers the same surplus as in the optimal pricing mechanism with ticket sales to passengers with signals above $\theta^*$. Hence it delivers the airline the same profit.

**Proof of Proposition 3**

Consider the following strategy profile. Ticketed passengers with value $v_j$ bid $v_j$. Unticketed passengers with value $v_k$ bid $v_j(v_k)$. To show that this is a dominant-strategy equilibrium, first consider a ticketed passenger with value $v_j$ when the profile of other bids is $b_{-j}$. Among those bids that exceed $R$, let $b^q_{-j}$ denote the $q$th highest bid, set equal to $R$ in case the number of such bids is smaller than $n$. If $j$’s bid $b_j$ exceeds $b^q_{-j}$, then $j$ will be seated and receive no transfer. If he bids less than $b^q_{-j}$, he will volunteer his seat and receive compensation in the amount of $b^q = b^q_{-j}$. Passenger $j$ wishes to volunteer his seat at this price if and only if $v_j \leq b^q_{-j}$. Thus by bidding according to the specified strategy profile, i.e. $b_j = v_j$, he volunteers his seat exactly when it is optimal to do so.

Next consider an unticketed passenger with value $v_k$. Let $b^q_{-k}$ denote the $q$th highest bid among the other passengers whose bid exceeds $R$, with $b^q_{-k}$ equaling $R$ if there are fewer than $q$. If $k$’s bid $b_k$ exceeds $b^q_{-k}$, then $k$ will win a seat and pay $v_k(b^q_{-k})$. If he bids
less than $b^g_{-k}$, he will lose and pay nothing. It is in the interest of passenger $k$ to fly and pay $v_k(b^g_{-k})$ if and only if $v_k \geq v_k(b^g_{-k})$. By bidding $b_k = v_j(v_k)$ as dictated by the strategy profile, he flies if and only if $v_j(v_k) \geq b^g_{-k}$, i.e. if and only if $v_k \geq v_k(b^g_{-k})$, i.e. exactly when it is in his interest to do so.

Finally to show that this dominant-strategy equilibrium implements the optimal allocation of seats it is enough to notice that the ranking of bids is the same as the ranking of virtual surpluses (with bids below the reserve $R$ being only for values such that virtual surplus is negative). This follows directly from the construction, using Condition (i) of Proposition 2, which ensures virtual surpluses are monotone in values over the relevant range.  

\[ \Box \]

### B Unrestricted mechanism and multiple fare classes

In this Appendix we derive the unrestricted optimal mechanism and show how our analysis can be extended to multiple fare classes.

#### Unrestricted mechanism

Suppose for simplicity that $G(\cdot | \theta)$ has full support on $[v, \bar{v}]$, for all signals $\theta$ (the arguments below, extend more generally, however). With no restriction on the space of possible mechanisms, the airline finds it optimal to contract with all passengers arriving in the market at date zero. Without loss of optimality, we consider direct mechanisms where each passenger $i$ reports his initial signal $\theta_i$ in case he arrives at date zero and then reports his value $v_i$ in the second period. We can think of the report of the signal as the passenger choosing among multiple fare classes available at date zero. We denote the null report, where the passenger fails to report at date zero, by $\emptyset$.

Let $\Omega_M$ denote a direct mechanism, comprising allocations $q = (q_i)_{i=1}^n$ and transfers $t = (t_i)_{i=1}^n$ such that, for all $\theta = (\theta_i)_{i=1}^n$ and $v$, the probability each passenger is seated is given by $q_i(\theta, v)$ and the total payment by each payment over two periods is given by $t_i(\theta, v)$ (given the absence of discounting, the timing of payments does not affect payoffs; we could equivalently consider date-0 payments, i.e. prices for each fare class in a continuum, and compensation satisfying Proposition 1). The allocation of seats must satisfy the same feasibility constraint introduced above: no more passengers may be seated than available seats.

Denote by $W_{\Omega_M}(\theta_i)$ the expected payoff of a passenger with signal $\theta_i$ given the opportunity to participate in the mechanism $\Omega_M$ at date zero (but with no other information about his value or about the signals or values of other passengers). Recall the “independent-shock representation” in the proof of Lemma 1. By the envelope theorem
(see Pavan et al. (2013)), a necessary condition for the incentive compatibility of truthful reporting of signals is that for all $\theta_i \geq 0$,

$$W^{\Omega \lambda} (\theta_i) = W^{\Omega \lambda} (0) + \int_0^{\theta_i} \mathbb{E} \left[ \frac{\partial z(y, \hat{\varepsilon})}{\partial \theta_i} q_i (y, \hat{\theta}_{-i}, z(y, \hat{\varepsilon}_i), \hat{\nu}_{-i}) \right] dy.$$  

We now conjecture that the value of $W^{\Omega \lambda} (0)$ is determined by the value a passenger with signal zero can obtain by deviating and reporting only at date 1. A passenger $i$ arriving at date 1 and truthfully reporting his value $v_i$, given that other passengers report $\theta_{-i}$ and $v_{-i}$, earns a payoff

$$W^{\Omega \lambda} (\emptyset, \theta_{-i}, v_i, v_{-i}) = W^{\Omega \lambda} (\emptyset, \theta_{-i}, v_i) + \int_{\emptyset}^{v_i} q_i (\emptyset, \theta_{-i}, y, v_{-i}) dy.$$  

We may optimally choose $W^{\Omega \lambda} (\emptyset, \theta_{-i}, v_i, v_{-i}) = 0$ while ensuring individual rationality of a passenger participating for the first time at date 1. A necessary condition for date 0 participation by a passenger with the 0 signal is therefore

$$W^{\Omega \lambda} (0) \geq \mathbb{E} \left[ \int_{\emptyset}^{\hat{v}_i} q_i (\emptyset, \hat{\theta}_{-i}, y, \hat{\nu}_{-i}) dy | \hat{\theta}_i = 0 \right]$$

$$= \mathbb{E} \left[ \frac{1 - G (\hat{v}_i | 0)}{g (\hat{v}_i | 0)} q_i (\emptyset, \hat{\theta}_{-i}, \hat{v}_i, \hat{\nu}_{-i}) dy | \hat{\theta}_i = 0 \right]$$

$$= \mathbb{E} \left[ \frac{1 - G (\hat{v}_i | 0)}{g (\hat{v}_i | \emptyset)} q_i (\emptyset, \hat{\theta}_{-i}, \hat{v}_i, \hat{\nu}_{-i}) dy | \hat{\theta}_i = \emptyset \right].$$

We conjecture that this is the only relevant participation constraint at date 0 and therefore that the inequality must bind in an optimal mechanism. Taking expectations and integrating by parts, the airline’s expected profit is then equal to

$$\sum_{i=1}^{n} \left( (1 - f(\emptyset)) \mathbb{E} \left[ \frac{z (\hat{v}_i, \hat{\varepsilon}_i) - \frac{1 - F (\hat{v}_i)}{f (\hat{v}_i)} \frac{\partial z (\hat{\theta}_i, \hat{\varepsilon}_i)}{\partial \hat{v}_i}}{q_i (\emptyset, \hat{\theta}_{-i}, \hat{v}_i, \hat{\nu}_{-i})} | \hat{v}_i, \hat{\varepsilon}_i, \hat{\theta}_i \geq 0 \right] \right)$$

$$+ f(\emptyset) \mathbb{E} \left[ \left( \hat{v}_i - \frac{1 - G (\hat{v}_i | \emptyset)}{g (\hat{v}_i | \emptyset)} - \frac{1 - f(\emptyset)}{f(\emptyset)} \frac{1 - G (\hat{v}_i | 0)}{g (\hat{v}_i | 0)} q_i (\emptyset, \hat{\theta}_i, \hat{\varepsilon}_i) \right) | \hat{v}_i, \hat{\varepsilon}_i, \hat{\theta}_i = 0 \right].$$  

(26)

We can now conjecture an optimal allocation rule along the same lines as for the single fare class mechanisms. Let

$$VS (\theta_i, \varepsilon_i) = \begin{cases} z (\theta_i, \varepsilon_i) - \frac{1 - F (\theta_i)}{f (\theta_i)} \frac{\partial z (\theta_i, \varepsilon_i)}{\partial \theta_i} & \text{if } \theta_i \geq 0, \\ z (\theta_i, \varepsilon_i) - \frac{1 - G (\theta_i | \emptyset)}{g (\theta_i | \emptyset)} - \frac{1 - f(\emptyset)}{f(\emptyset)} \frac{1 - G (\theta_i | 0)}{g (\theta_i | 0)} & \text{if } \theta_i = \emptyset \end{cases}$$

(27)

Let $\mathbf{q}^*$ be the allocation which maximizes the virtual surplus $VS$. The following result
gives conditions under which this allocation is implementable by an appropriate system of transfers.

**Proposition 4.** Suppose that $VS(\cdot, \cdot)$ as defined by Eq. (27) is non-decreasing in both arguments. Then there exists a system of transfers $t^*$ that implements the allocation $q^*$. The mechanism $(q^*, t^*)$ is profit maximizing in the class of all possible mechanisms.

**Proof.** The payoffs $W_{\Omega M}(\theta, v)$ to be earned in equilibrium are pinned down by the allocation rule, the fact that the date 0 participation constraint binds at $\theta_i = 0$, and that a passenger arriving at date 1 with value $v$ expects zero surplus. We can define transfers simply by the identity $t_i(\theta, v) = q_i(\theta, v) \theta_i - W_{\Omega M}(\theta, v)$, so that passengers receive the intended payoffs provided they report truthfully. Using that each passenger’s probability of receiving the seat is non-decreasing in his value for it, the mechanism described above can be shown to be ex-post incentive compatible at date 1. Incentive compatibility of truthful reporting of the signal at date zero, given that the passenger participates at that date, also follows from the monotonicity of the allocation in both signals and values (see Pavan et al. (2013), for instance) and from the first-order stochastic dominance property of the stochastic process. What is left to check is that all passengers with signals in $[0, 1]$ prefer to participate at date 0 rather than delaying participation to date 1. This follows from essentially the same argument as in the proof of Lemma 1, after noting that, for each reported value of a passenger $i$ (holding fixed the reports of the other passengers) the probability that passenger receiving the good, for each possible value $v_i$, is lower if he delays participation until date 1. The virtual surplus for the unrestricted mechanism is closely related to those for the single fare-class mechanism studied above. The airline would like to distort downwards the probability a passenger with a low signal is seated so as to reduce the rents that must be left to passengers with higher signals to dissuade them from mimicking the distribution of reports by passengers with the lower signals. This idea is familiar from the other work on dynamic mechanism design. With the single fare-class mechanism studied above, the airline lacks the flexibility to distinguish the allocations for passengers with different signals, but it faces an incentive to achieve a similar end by selling tickets only to passengers with high signals (conferring a higher probability of being seated on the holder).

The airline would also like to reduce the probability of flying for passengers who only arrive to the market at date 1 in order to reduce the rents available to them. The advantage in doing so is that it permits the airline to reduce the rent that must be left to passengers who arrive at date zero in order to persuade them to participate at that date. This again parallels the airline’s incentive to limit the rents of non-ticket holders in the
single fare class mechanism.

The relationship between the virtual surplus $\text{VS} (\cdot, \cdot)$ for the unrestricted mechanism and the virtual surpluses of the single fare class mechanism $\overline{\text{VS}} (\cdot)$ and $\underline{\text{VS}} (\cdot)$ for ticketed and unticketed passengers turns out to be simple. Given the threshold for ticket purchases $\theta^*$, for each $v_i$, we have

$$\overline{\text{VS}} (v_i) = \mathbb{E} \left[ \text{VS} \left( \tilde{\theta}_i, \tilde{v}_i \right) \mid \tilde{\theta}_i \geq \theta^*, \tilde{v}_i = v_i \right]$$

while

$$\underline{\text{VS}} (v_i) = \mathbb{E} \left[ \text{VS} \left( \tilde{\theta}_i, \tilde{v}_i \right) \mid \tilde{\theta}_i < \theta^*, \tilde{v}_i = v_i \right].$$

**Multiple fare classes**

The above suggests how our analysis can be extended to allow the airline to use two or more fare classes. If $L$ fare classes are permitted, then we consider pricing mechanisms with thresholds $\theta^*_1, \ldots, \theta^*_L \in [0, 1]$, ordered from highest to lowest, such that all passengers with signals in $I_1 = [\theta^*_1, 1]$ purchase class-1 tickets, those with signals in $I_2 = [\theta^*_2, \theta^*_1)$ purchase class-2 tickets, and so forth, with passengers with signals in $I_{L+1} = (-\infty, \theta^*_L)$ not purchasing any ticket. Assuming that a passenger in a higher class is favored in the seating allocation in the sense introduced in Section 3, with all ticketed passengers favored over unticketed passengers, any pricing mechanism with $L$ classes has this threshold property.

We can then conjecture an allocation of seats to passengers with the highest non-negative virtual surpluses, where these virtual surpluses are given, for each fare class $l$ and value $v_i$, by $\mathbb{E} \left[ \text{VS} \left( \tilde{\theta}_i, \tilde{v}_i \right) \mid \tilde{\theta}_i \in I_l, \tilde{v}_i = v_i \right]$. We can then look for conditions guaranteeing that these allocations are monotone in $v_i$ for each fare class (and for unticketed passengers), and which favor passengers in a higher fare class conditional on having the same value $v_i$. Such conditions ensure the existence of ticket prices and compensation rule which implements the proposed allocations.