Penny Auctions*

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Abstract

I study penny auctions, a novel auction format where every bid increases the price by a small amount but placing a bid is costly. Outcomes of real-life penny auctions are surprising. Even when selling cash, the seller can get revenue that is much higher or lower than its nominal value, and losers in an auction sometimes pay much more than the winner. In this article, I characterize all symmetric Markov-perfect equilibria of penny auctions and study penny auctions' properties. I conclude that the high variance of outcomes is a natural property of the penny auction format.

JEL: D11, D44, C73

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1 Introduction

A typical penny auction sells a new brand-name gadget at a starting price of zero dollars and sets a timer at one minute. When the auction starts, the timer starts to tick down, and players may submit bids. Each bid costs one dollar, increases the price by one cent, and resets the timer to one minute. Once the timer reaches zero, the last bidder is declared the winner and can purchase the object at the final price.

Penny auctions and English auctions have similar structures with one significant difference: in penny auctions, bidders have to pay bid fees for each price increment.

Penny auctions have surprising properties in practice. (For detailed analysis and description, see appendix A) First, the relation between the final price and the value of the object is stochastic, has a high mass on very low values, and has a long tail on high values. Second, the winner of the auction usually pays less than the value of the object. However, because the losers collectively pay large amounts, the revenue to the seller is often higher than the value of the object. In fact, the losers sometimes pay more than the winner.

Penny auctions are interesting for several reasons. First, penny auctions are popular in real life, and they are not a special case of any well-known auction format, so it is important to know their properties. In this article, I characterize all symmetric Markov-perfect equilibria (SMPE) in penny auctions and show that, even under standard assumptions (fully rational, risk-neutral bidders; common value of the prize; and common knowledge of the number of players), equilibria have most of the characteristics described.

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above. Second, the popularity of penny auctions has caught the attention of policymakers around the world who are asking whether they might be scams or games of chance. In this article, I show that, although penny auctions do not use any randomization devices, equilibria in penny auctions involve mixed strategies. Thus, from the individual bidder’s perspective, the penny auction format is similar to lotteries'. Third, I argue that high revenues in penny auctions cannot be explained by fully rational behavior; therefore, penny auctions provide valuable empirical evidence for testing theories in behavioral economics.

This article is one of the first to study penny auctions with two others being Augenblick (2012) and Platt, Price, and Tappen (2013). In this article, I make three contributions to the literature. First, I show how to characterize SMPE in penny auctions. Second, I characterize all SMPE in penny auctions under two different assumptions: the simultaneous-bids assumption and the single-bid assumption. Most studies use the single-bid assumption and only focus on one particular equilibrium. Augenblick (2012) and Platt, Price, and Tappen (2013) focus on empirical analysis of penny auctions. To that end, they employ a particularly tractable specification that can be used in empirical analyses and extended to accommodate non-standard preferences. Third, full characterization allows me to compare equilibria under different assumptions and make conclusions about the auction format itself rather than a particular equilibrium.

In this article, I show that penny auctions with fully rational, risk-neutral agents cannot explain the seller’s high average profit. This fact has inspired subsequent literature to extend the model with more complex preferences. Because the auction format leads to highly uncertain outcomes, risk-loving individuals would be happy to pay more than the expected value of winning. Platt, Price, and Tappen (2013) finds some evidence for this. Gnutzmann (2014) extends the analysis to cumulative-prospect-theory preferences and argues that it explains the behavior even better than standard risk-loving behavior. Augenblick (2012) proposes that the explanation could be another behavioral bias—the sunk-cost bias. Caldara (2012) used lab experiments and found that high revenues came mainly from the agents who are inexperienced and not strategically sophisticated.

There are other forms of pay-to-bid auctions than penny auctions. As in penny auctions, most revenue comes from bid fees rather than the winning price. A price-reveal auction is a descending-price auction in which the current price is hidden and bidders can privately observe the price for a fee. Gallice (2012) shows that, under the standard assumptions, those auctions would end quickly and would be unprofitable. In unique-price auctions, bidders submit positive integers as bids, and the winner is the one who submitted the lowest or highest unique number. Raviv and Virag (2009) and Östling, Wang, Chou, and Camerer (2011) have found a surprising degree of convergence toward the equilibrium in these auctions.

The idea of penny auctions is similar to the dollar auction introduced in Shubik (1971). In this kind of auction, the auctioneer sells cash to the highest bidder, but the two highest bidders pay their bids. Shubik uses dollar auctions to illustrate potential weaknesses of traditional solution concepts and describes this kind of auction as extremely simple, highly amusing, and usually highly profitable for the auctioneer.

The dollar auction is a kind of all-pay auction that has been used to model rent-seeking, research and development races, political contests, and job promotions. Baye, Kovenock, and de Vries (1996) provides full characterization of equilibria under full information in one-shot (first-price), all-pay auctions. The second-price, all-pay auction (also called a war of attrition) has been used to study evolutionary stability of conflicts, price wars, bargaining, and patent competition. Full characterization of equilibria under full

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1The Better Business Bureau (BBB) called penny auctions a scam and said that “BBB recommends you treat them the same way you would legal gambling in a casino—know exactly how the bidding works, set a limit for yourself, and be prepared to walk away before you go over that limit” (BBB, 2012).

2The first versions of all three articles were written independently in 2009.
information is given by Hendricks, Weiss, and Wilson (1988). Although a penny auction is an all-pay auction, it is not a special case of previously documented auction formats because, in contrast to standard all-pay auctions, the winner of a penny auction might pay less than the losers.

The article is organized as follows: section 2 introduces a theoretical model and discusses its assumptions; section 3 analyzes the condition that the price increment is zero and, therefore, the auction game is infinite; section 4 discusses the condition that the price increment is strictly positive; and section 5 concludes.

2 Model

The auctioneer sells an object with common value $V$. There are $N + 1 \geq 3$ players (bidders) participating in the auction denoted by $i \in \{0, 1, \ldots, N\}$. All bidders are risk-neutral and at each point of time maximize the expected continuation value of the game.

The auction is dynamic, and players submit bids at discrete time points $t \in \{0, 1, \ldots\}$. The auction starts at initial price $P_0$. At time $t = 0$, all $N + 1$ bidders are non-leaders. At each period $t > 0$, exactly one of the players is the current leader, and the other $N$ players are non-leaders.

In each period $t$, non-leaders simultaneously choose whether to submit a bid or pass. Each submission of a bid costs $C$ dollars and increments the price by $\varepsilon$. If $K > 0$ of the non-leaders submit a bid, each of them has equal probability, $1/K$, to be the leader in the next period. In modeling simultaneous bids by multiple agents, I consider two cases:

1. Simultaneous-bids case: When $K > 0$ players choose to bid, all $K$ are accepted so that the price increases according to $P_{t+1} = P_t + \varepsilon K$, and each of these $K$ players pays $C$ to the seller.

2. Single-bid case: When $K > 0$ players choose to bid, only one bid is accepted so that the price increases according to $P_{t+1} = P_t + \varepsilon$, and only the new leader pays $C$ to the seller.

If all non-leaders pass at time $t$, the auction ends. If the auction ends at $t = 0$, then the seller keeps the object, and if it ends at $t > 0$, then the current leader can purchase the object at price $P_t$. Finally, if the game never ends, all bidders get payoffs $-\infty$, and the seller keeps the object. All the parameters of the game are commonly known, and the players know the current leader and observe all previous bids by all players.

I use the following normalizations. In case $\varepsilon > 0$, I normalize $v = \frac{V - P_0}{\varepsilon}$, $c = \frac{C}{\varepsilon}$, and $p_t = \frac{P_t - P_0}{\varepsilon}$. In case $\varepsilon = 0$, I use $v = V - P_0$, $c = C$, and $p_t = P_t - P_0$. Therefore, in all cases, $p_0 = 0$. Given the assumption and normalizations, a penny auction is fully characterized by $(N, v, c, \varepsilon)$, where $\varepsilon$ is only used to distinguish between the zero-price-increment and positive-price-increment cases.

Finally, I assume that $v - c > 1$ because, otherwise, the auction would never start.

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3When the number of bidders is either one or two, the game differs significantly because there are no simultaneous bids after period zero. When $N + 1 \leq 2$, it is straightforward to verify that the game ends after at most one round of bids in all equilibria.

4In real penny auctions, players can submit bids in real time, and all bids are accepted. The single-bid case captures a limit where the players can instantly respond to opponents’ bids, and the simultaneous-bids case captures the situation where nearly simultaneous bids are all accepted.

5This is the case that most of the literature makes, starting with Augenblick (2012) and Platt, Price, and Tappen (2013). Note that they also assume that $C$ is a multiple of $\varepsilon$, but I do not.

6Clearly, if the final price $P_t > V$, the winner will not purchase the object.
Symmetric Markov-Perfect Equilibria

I focus on SMPE, which impose two requirements in addition to the standard subgame-perfect Nash equilibrium assumptions. The first is symmetry, which means that the players’ identities do not play any role. This assumptions fits because players are anonymous in all real penny auctions. Second, I look for equilibria in Markovian strategies. Players using Markovian strategies only condition their behavior on the current price and the number of active bidders; they do not condition their behavior on the whole history of bids or on identities of leaders. In real penny auctions, the only information revealed to buyers is the identity of the leader and the current price, so this restriction also matches real-life penny auctions.

Note that this restriction still allows players to deviate to non-Markovian strategies.

Focusing on SMPE reduces the set of equilibria as well as simplifies the notation and construction of equilibria. Typically, constructing equilibria would require using general strategy profiles. As proposition 2.1 shows, any equilibrium can be represented by a vector \( q \), which can be found by finding a stage-game Nash equilibrium for each state sequentially.

**Proposition 2.1.** A strategy profile is an SMPE if and only if it can be represented by \( q \), where \( q(\cdot) \) is the Nash equilibrium of the stage-game in a particular state, taking into account the continuation values.

The proof for proposition 2.1 is in appendix B.

In case \( \varepsilon > 0 \), the set of states will be the set of all possible prices. Therefore, we can use the current price \( p \) (independent of time or history) as the current state variable and solve for a symmetric Nash equilibrium in this state, given the continuation values at states that follow each profile of actions. So the equilibrium is fully characterized by a \( q : \mathbb{Z}_+ \to [0, 1] \), where \( q(p) \) is the probability of submitting a bid that each non-leader independently uses at price \( p \).

In case \( \varepsilon = 0 \), the equilibrium characterization is even simpler because there are only two states. In the beginning of the game there are \( N + 1 \) non-leaders, and in any of the following histories, the number of non-leaders is \( N \). So the equilibrium is characterized by \( (\hat{q}_0, \hat{q}) \), where \( \hat{q}_0 \) is the probability that a player submits a bid at period 0 and \( \hat{q} \) is the probability that a non-leader submits a bid in any of the following periods. The SMPE can be found simply by solving for Nash equilibria at both states, taking into account the continuation values.

### 3 Auctions with Zero Price Increment

When the price increment is zero (\( \varepsilon = 0 \)), the auction is similar to an infinitely repeated game. After each round of bids, bid costs are already sunk, and the payoffs for winning are the same.

By proposition 2.1, any SMPE is characterized by a pair \( (\hat{q}_0, \hat{q}) \), where \( \hat{q}_0 \) is the probability that a non-leader will submit at round 0 and \( \hat{q} \) is the probability that a non-leader submits a bid in any subsequent round. Let \( \hat{v}^* \) and \( \hat{v} \) be the leader’s and non-leaders’ continuation values (after round zero), respectively. Proposition 3.1 shows that the SMPE is unique and fully characterizes this equilibrium.

**Proposition 3.1.** When \( \varepsilon = 0 \), SMPE \( (\hat{q}_0, \hat{q}) \) is such that \( \hat{q} \in (0, 1) \) is uniquely determined by the equality

\[
(1 - \hat{q})^N \Psi_N(\hat{q}) = \frac{c}{\hat{v}},
\]

and

(i) in the simultaneous-bids case, \( \hat{q}_0 \in (0, 1) \) is uniquely defined by

\[
(1 - \hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{\hat{v}};
\]

(ii) in the single-bid case, there is a continuum of equilibria, one for each \( \hat{q}_0 \in [0, 1] \).

\[\text{In contrast, when \( \varepsilon > 0 \), the game always ends in finite time as shown in lemma 4.1 in section 4.}\]
where

\[ \Psi_K(q) = \begin{cases} 
  \frac{1-(1-q)^K}{qK} & \text{in the simultaneous-bids case,} \\
  1 & \text{in the single-bid case.}
\end{cases} \]

**Proof** First, notice that there is no (symmetric) pure-strategy equilibria in this game. If \( \hat{q} = 1 \), the game never ends, and all players get \(-\infty\), an outcome which cannot be an equilibrium. If \( \hat{q} = 0 \), then \( \hat{v}^* = v \), and \( \hat{v} = 0 \). This outcome cannot be an equilibrium because a non-leader would want to deviate and submit a bid to get \( \hat{v}^* - c \), which is higher than \( \hat{v} \) (by assumption, \( v > c + 1 > c \)). Therefore, in any equilibrium, \( \hat{q} \in (0, 1) \).

In any period, a non-leader’s expected value is equal to the expected value from not submitting a bid. The other \( N-1 \) non-leaders submit a bid each with probability \( \hat{q} \), so the game ends with probability \( (1-\hat{q})^{N-1} \) and continues from the same point with probability \( 1 - (1-\hat{q})^{N-1} \). Therefore,

\[ \hat{v} = [1 - (1-\hat{q})^{N-1}]\hat{v} + (1-\hat{q})^{N-1}0 \iff \hat{v} = 0 \]

because \( 0 < \hat{q} < 1 \). Consequently, the leader has \( (1-\hat{q})^N \) chance to win the object; otherwise, the leader becomes a non-leader who gets \( \hat{v} = 0 \). Therefore,

\[ \hat{v}^* = (1-\hat{q})^N v + [1 - (1-\hat{q})^N]\hat{v} = (1-\hat{q})^N v. \]

The value of \( \hat{q} \) is pinned down by the mixing condition of a non-leader,

\[ \hat{v} = 0 = \sum_{K=0}^{N-1} \binom{N-1}{K}\hat{q}^K (1-\hat{q})^{N-1-K} \left[ \frac{1}{K+1} \hat{v}^* + \frac{K}{K+1} \hat{v} \right] - c \iff \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}), \]

where the function \( \Psi_N(q) \) is player \( i \)'s probability of becoming the new leader after submitting a bid when \( N-1 \) other non-leaders submit their bids independently, each with probability \( q \). Additionally, because

\[ \Psi_N(q) = \sum_{K=0}^{N-1} \binom{N-1}{K}q^K (1-q)^{N-(K+1)} \frac{1}{K+1} = \frac{1-(1-q)^N}{qN}, \]

it is straightforward to verify that \( \Psi_N(q) \) is a strictly decreasing continuous function with limits 1 and \( \frac{1}{N} \) as \( \hat{q} \to 0 \) and \( \hat{q} \to 1 \), respectively. As \( \hat{q} \) changes in \((0,1)\), it takes each value in the interval \((\frac{1}{N}, 1)\) exactly once. The function \((1-\hat{q})^N \) is also a strictly decreasing continuous function with limits 1 and 0, so the function \((1-\hat{q})^N \Psi_N(q) \) is a strictly decreasing continuous function in \( \hat{q} \) and takes all values in the interval \((0,1)\). Therefore there exists unique \( \hat{q} \in (0,1) \) which solves the equation \((1-\hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v} \in (0,1) \).

Next, I consider period 0 to find the equilibrium strategy at \( \hat{q}_0 \). I denote the expected value that a player gets from playing the game by \( \hat{v}_0 \), and I claim that \( \hat{q}_0 \in (0,1) \). Suppose first that \( \hat{q}_0 = 0 \), a condition that would end the game instantly with all bidders getting nothing. By submitting a bid, a player could ensure becoming the leader with certainty in the next round and therefore getting value \( \hat{v}^* - c = (1-\hat{q})^N v - c \). The equilibrium condition says that this value must be less than the equilibrium payoff of zero; however, in that case,

\[ (1-\hat{q})^N v - c \leq 0 \iff (1-\hat{q})^N \leq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}), \]

so \( \Psi_N(\hat{q}) \geq 1 \). There is a contradiction because \( \Psi_N(\hat{q}) < 1 \) for all \( \hat{q} > 0 \).

Next, suppose that \( \hat{q}_0 = 1 \) is an equilibrium so that each bidder must weakly prefer bidding over
not bidding and consequently getting the continuation value of a non-leader, \( \hat{v} = 0 \). The equilibrium condition is

\[
\frac{1}{N+1} \hat{v}^* - c = \frac{1}{N+1} (1 - \hat{q})^N v - c \geq 0 \iff (1 - \hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}
\]

so \( \Psi_N(\hat{q}) \leq \frac{1}{N+1} \frac{1}{N} \), which is also a contradiction.

Thus, in equilibrium, \( 0 < \hat{q}_0 < 1 \) is defined by

\[
0 = \sum_{K=0}^{N} \binom{N}{K} \hat{q}_0^K (1 - \hat{q}_0)^{N-K} \frac{1}{K+1} \hat{v}^* - c \iff (1 - \hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v}
\]

The following equivalent form of the equation shows that the previous equation defines \( \hat{q}_0 \) uniquely for a fixed \( \hat{q} \in (0, 1) \):

\[
(1 - \hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v} \iff \Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q}).
\]

I have already shown that \( \Psi_N(\hat{q}) \in \left( \frac{1}{N+1}, 1 \right) \). As argued above, (continuous, strictly decreasing) \( \Psi_{N+1}(\hat{q}_0) \) takes values in the interval \( \left( \frac{1}{N+1}, 1 \right) \supset \left( \frac{1}{N}, 1 \right) \), so the equation must have a unique solution \( \hat{q}_0 \).

Proof for the single-bid case is analogous for the most part with the exception that \( c \) must be paid only by the new leader, so the mixing condition for a non-leader is

\[
\hat{v} = 0 = \sum_{K=0}^{N-1} q^K (1 - q)^{N-1-K} \left[ \frac{1}{K+1} [\hat{v}^* - c] + \frac{K}{K+1} \hat{v} \right] \iff c = \hat{v}^* = (1 - q)^N v
\]

so that \( \hat{v} \) is defined by the condition \( (1 - \hat{v})^N = \frac{c}{v} \). Similarly, \( \hat{v}_0 \) is defined by the equation \( (1 - \hat{v}_0)^{N+1} = \frac{c}{v} \), both sides of which are special cases of the expressions given in the proposition with \( \Psi_K(q) = 1 \).

Consequently, in period 0, buyers are always indifferent between bidding and not bidding because both give zero payoff under any realization of uncertainty. Therefore, each \( \hat{q}_0 \in [0, 1] \) can be an equilibrium strategy.

In the single-bid case, the set of equilibria characterized in proposition [3.1] coincides with equilibria in Augenblick (2012) and Platt, Price, and Tappen (2013), but my result is more general in two ways. First, Augenblick and Platt, Price, and Tappen verify that these strategy profiles are SMPE, but proposition [3.1] additionally shows that there are no other SMPE. Second, it also characterizes SMPE for the simultaneous-bids case and thus makes it possible to form conclusions about the robustness of the single-bid assumption [5].

Corollary [3.2] illustrates differences between the equilibria in (fixed-price) penny auctions in the simultaneous-bids case and the single-bid case. First, in the simultaneous-bids case, the equilibrium is unique, and in particular, bidding with certainty in period 0 is not an equilibrium. Second, the probability that at least one player bids in period \( t > 0 \) is a strictly decreasing function of the number of bidders, \( N \). This characteristic arises because the increase in the number of bidders has two effects on the decision to submit a bid: first, for a fixed \( \hat{q} \), increasing bidders decreases the probability that the game ends in the next period, and second, increasing bidders decreases the probability of becoming the next leader after submitting a bid. The latter effect is missing in the single-bid case because only the next
leader incurs the bid cost. The lack of this second effect simplifies the analysis significantly because the continuation probability is simply equal to $\xi$. Augenblick (2012) and Platt, Price, and Tappen (2013) cleverly exploit this simplification in their empirical analysis and extensions. Corollary 3.2, however, emphasizes that their analysis is not robust to changes in the modeling of simultaneous bids.

**Corollary 3.2.** When $\varepsilon = 0$, penny auctions have the following properties:

(i) Simultaneous-bids case

(a) At any period $t > 0$, the probability that an individual bidder submits a bid is constant $\hat{q} \in (0, 1)$, and the probability that at least one bidder submits a bid is constant $\hat{Q} = 1 - (1 - \hat{q})^N \in (0, 1)$. Both $\hat{q}$ and $\hat{Q}$ are strictly decreasing in $N$ and $\xi$.

(b) The probability that an individual bidder submits a bid at period 0 is $\hat{q}_0 \in (0, 1)$, and the probability that at least one bidder submits a bid (or equivalently the probability of selling the object) is $\hat{Q}_0 = 1 - (1 - \hat{q})^N \in (0, 1)$. Both the individual probability, $\hat{q}_0$, and the cumulative probability, $\hat{Q}_0$, are strictly decreasing in $\xi$.

(ii) Single-bid case

(a) At any period $t > 0$, the probability that an individual bidder submits a bid is constant $\hat{q} \in (0, 1)$, and the probability that at least one bidder submits a bid is constant $\hat{Q} = 1 - (1 - \hat{q})^N \in (0, 1)$. The individual probabilities $\hat{q}$ and $\hat{Q}$ are strictly decreasing in $N$ and $\xi$; the cumulative probability $\hat{Q}$ is strictly decreasing in $\xi$ and independent of $N$.

(b) There is a continuum of equilibria, one for each $\hat{q}_0 \in [0, 1]$, where $\hat{q}_0$ is the probability that an individual bidder submits a bid in period 0. The probability that at least one bidder submits a bid (the probability of selling the object), $\hat{Q}_0$, can therefore be any number in $[0, 1]$.

(iii) The expected value to players is 0.

(iv) The expected revenue to the seller, conditional on sale, is $v$.

**Proof**

(i) Simultaneous-bids case

(a) The proposition showed that the probability that an individual bidder submits a bid at $t > 0$ is a constant $\hat{q} \in (0, 1)$; therefore, the probability that any bidder submits a bid is also constant, $\hat{Q} = 1 - (1 - \hat{q})^N \in (0, 1)$.

For comparative statics, note that

$$\Psi_N(q) = \frac{1 - (1 - q)^N}{[1 - (1 - q)]^N} = \frac{1 + (1 - q) + (1 - q)^2 + \cdots + (1 - q)^{N-1}}{N},$$

which is decreasing both in $q$ and in $N$. Also, $(1 - q)^N$ is decreasing both in $q$ and in $N$; therefore, $(1 - q)^N \Psi_N(q)$ is decreasing in both arguments. Because $\hat{q}$ is defined by $(1 - q)^N \Psi_N(q) = \xi$, $\hat{q}$ is decreasing in $N$ and $\xi$. Because $\hat{Q} = 1 - (1 - \hat{q})^N$ is increasing in $\hat{q}$, $\hat{Q}$ is decreasing in $\xi$, as well.

Finally, $\hat{Q}$ is decreasing in $N$ as shown by rewriting the definition of $\hat{q}$ in terms of $\hat{Q}$ as

$$\frac{\hat{Q}}{N[1 - (1 - \hat{Q})^N \frac{\hat{Q}}{N}]} = \frac{c}{v},$$

7
It suffices to show that the left-hand side is strictly increasing in $\hat{Q} \in (0, 1)$ and strictly decreasing in $N \geq 2$.

Because the expression was decreasing in $\hat{q}$, it is also decreasing in $\hat{Q}$. To prove that it is decreasing in $N$, I show that the denominator, $N[1 - (1 - \hat{Q})^\frac{1}{N+1}]$, is increasing in $N$. Fix $N \geq 2$ and $\hat{Q} \in (0, 1)$, and define $x = (1 - \hat{Q})^\frac{1}{N+1}$. Then $x \in (0, 1)$, and I need to show that

$$(N + 1) \left( 1 - (1 - \hat{Q})^\frac{1}{N+1} \right) = (N + 1) \left( 1 - x^N \right) > N \left( 1 - x^{N+1} \right) = N \left( 1 - (1 - \hat{Q})^\frac{1}{N+1} \right),$$

which can be rewritten as

$$N x^N (1 - x) < 1 - x^N \iff x^N < \frac{1 - x^N}{N(1 - x)} = \frac{1 + x + x^2 + \ldots + x^{N-1}}{N}.$$

This equation is satisfied due to the following: for all $x \in (0, 1)$ and $N \geq 2$, $x^K > x^N$ for all $K < N$. Therefore, $\hat{Q}$ is increasing in $N$.

(b) If $N + 1 > 2$, then by the proposition, $\hat{q}_0 \in (0, 1)$; therefore, $\hat{Q}_0 = 1 - (1 - \hat{q})^N \in (0, 1)$. Because $\Psi_N(\hat{q}_0) = \Psi_N(\hat{q})$ and $\Psi_N(q)$ is decreasing both in $q$ and in $N$, $\hat{q}_0$ is strictly decreasing in $\frac{c}{v}$ and $\hat{q}_0 < \hat{q}$. This result also implies that $\hat{Q}_0$ is strictly decreasing in $\frac{c}{v}$.

(ii) Single-bid case

(a) This proof is analogous to the proof for the simultaneous-bids case except for the relationship between $\hat{Q}$ and $N$. Note that $\hat{q}$ is defined by $(1 - \hat{q})^N = \frac{c}{v}$ so that $\hat{Q} = 1 - \frac{c}{v}$, which is independent of $N$.

(b) Any $\hat{q}_0 \in [0, 1]$ can be an equilibrium strategy by the proposition. It follows that $\hat{Q}_0 = 1 - (1 - \hat{q}_0)^N$ can take any values in $[0, 1]$, as well.

(iii) In both the simultaneous-bids case and the single-bid case, bidders are indifferent between bidding and not bidding in each period so that one strategy over which they randomize is never bidding—a strategy that gives a value of zero.

(iv) There is another way to compute the ex-ante value to players, $\hat{v}_0$. Let the actual number of bids the players submitted in a particular realization of uncertainty be $B$. Conditioning on sale means that $B > 0$. Let $Pr(B | B > 0)$ denote the conditional probability that the number of bids is $B$. Because the value to the winner is $v$ and all the players collectively paid $B \cdot c$ in bid costs, the aggregate value to the players is $v - Bc$. Due to symmetry and risk-neutrality, this value is ex-ante divided equally among all players, so

$$0 = \hat{v}_0 = \frac{1}{N + 1} \sum_{B=1}^{\infty} [v - Bc] Pr(B | B > 0) = \frac{v - cE(B | B > 0)}{N + 1}.$$

Revenue to the seller, given that the object is sold, is $Bc$. Therefore, the expected revenue is $R = E(Bc | B > 0) = cE(B | B > 0) = v$. 

Corollary 3.3 illustrates that, although all the payoffs are precisely determined in expected terms, any outcome of the game can occur with strictly positive probability in actual realizations. All three
implications in the corollary follow from the simple fact that, in equilibrium in each period \( t > 0 \), the probability that there are more bids \( \hat{Q} \in (0, 1) \).

**Corollary 3.3.**

(i) With positive probability, the seller sells the object after just one bid and gets revenue \( c \). The winner gets \( v - c \), and the losers pay nothing.

(ii) On the other hand, there is a positive probability that revenue is larger than any fixed number \( M \). Moreover, with positive probability, the revenue is bigger than \( M \), and the winner only pays \( c \).

4 Auctions with Positive Price Increments

In this section, I analyze penny auctions with positive price increments. Following proposition 2.1, any SMPE can be characterized by a vector \( \mathbf{q} = (q(0), q(1), \ldots) \), where \( q(p) \) is non-leaders’ probability to bid at price \( p \). It is both necessary and sufficient to check for stage-game Nash equilibria, given the continuation payoffs induced by the chosen actions. For a given equilibrium, I denote the leader’s continuation value by \( v^*(p) \) and non-leaders’ continuation value by \( v(p) \).

With a positive price increment, the price increases over time. Therefore, the game must end in finite time—at least when the price rises above the value of the object. Lemma 4.1 establishes this observation formally and gives an upper bound to the prices at which bidders will still be active.

**Lemma 4.1.** Suppose \( \varepsilon > 0 \), and fix any SMPE. Then, \( q(p) = 0 \) for all \( p \geq \lfloor v - c \rfloor \), where \( \lfloor x \rfloor = \max \{ k \in \mathbb{Z} : x \leq k \} \) is the floor function.

**Proof** If \( p > v \), the upper bound of the winner’s payoff in this game is \( v - p < 0 \), so any continuation of this game is worse for all the players. Therefore, prices for which \( q(p) > 0 \) are bounded by \( v \).

Let \( \hat{p} \) be the highest price for which \( q(\hat{p}) > 0 \). Suppose by contradiction that \( \hat{p} \geq \lfloor v - c \rfloor \). Because \( q(\hat{p} + K) = 0 \) for all \( K \in \mathbb{N} \), the game ends instantly if the price rises above \( \hat{p} \). Therefore, \( v(\hat{p} + K) = 0 \), and

\[
v^*(\hat{p} + K) = v - (\hat{p} + K) \leq (v - c) - \lfloor v - c \rfloor - K + c < c,
\]

because by definition \( (v - c) - \lfloor v - c \rfloor < 1 \); thus, \( (v - c) - \lfloor v - c \rfloor - K < 0 \) for all \( K \in \mathbb{N} \).

Therefore, submitting a bid with positive probability at \( \hat{p} \) gives an expected payoff that is strictly lower than the cost of submitting the bid, \( c \), which is a contradiction.

Because the general characterization covers many cases, I use a simple example with three players to make the characterization more concrete.

**Simple Example**

Suppose the value of the object is \( v = 3 \), the bid cost and bid increment are both \( c = \varepsilon = 1 \), and the number of bidders is \( N + 1 = 3 \). By lemma 4.1, for any \( p \geq \lfloor v - c \rfloor = 2 \), any SMPE must have \( q(p) = 0 \). Thus, \( v^*(2) = 1, v(2) = 0 \), and \( v^*(p) = v(p) = 0 \) for all \( p \geq 2 \).
Simultaneous-bids Case

At price $p = 1$, submitting a bid would cost 1 with certainty and would give the player $v^*(2) = 1$ only if the other non-leader does not bid. Therefore, in SMPE, $q(1) = 0$; thus, $v^*(1) = 2$, and $v(1) = 0$. Finally, at $p = 0$, the action must be mixed, in particular, such that

$$[1 - q(0)]^2 v^*(1) + 2q(0)[1 - q(0)]rac{v^*(1)}{2} - c = 0 \Rightarrow q(0) = \frac{3 - \sqrt{5}}{2} \approx 0.3820.$$ 

In this equilibrium, the value to bidders is clearly $v(0) = 0$, probability of selling the object is $1 - [1 - q(0)]^3 \approx 0.7639 > 0$, and the expected revenue condition on selling is equal to $v = 3$. The ex-ante probability that the game ends at any price $p \in \{0, 3\}$ is strictly positive.

Finally, note that this analysis is essentially unchanged if the value of the object is $v + \gamma$ for a small $\gamma > 0$. In this case, $q(1)$ will be a small positive number, and $q(0)$ will be slightly higher, but when $\gamma$ is close to 0, the equilibrium will converge to the equilibrium described above.

Single-bid case

First, assume that $v = 3$. At $p = 1$, non-leaders are indifferent between bidding and not bidding because when they become the leader, they will get value $v^*(2) = 1$ but pay a bid cost of 1. Therefore, any $q(1) \in [0, 1]$ can be part of an equilibrium strategy. For a given $q(1)$, $v^*(1) = [1 - q(1)]^2/2$, and $v(1) = 0$. Because $q(1)$ affects the continuation value, it is useful to define the critical value $\bar{q}$ at which the leader’s value would be exactly equal to bid cost; that is, $[1 - \bar{q}]^2/2 = 1$, which gives $\bar{q} = 1 - \frac{1}{\sqrt{2}} \approx 0.2929$.

Depending on $q(1)$, there are three cases at $p = 0$.

1. When $q(1) > \bar{q}$, the continuation value is low, and the dominant strategy at price $p = 0$ is not to bid; that is, $q(0) = 0$. The value to bidders is $v(0) = 0$, and the probability of sale is 0.

2. When $q(1) < \bar{q}$, the continuation value is high, and the dominant strategy at price $p = 1$ is to bid with certainty; that is, $q(1) = 1$. The value to bidders is $v(0) = \frac{2}{3}[1 - q(1)]^2 \in [0, \frac{1}{3}]$, the object is sold with certainty, and the expected revenue is $4 - 2[1 - q(1)]^2 \in (2, 3)$. The ending price of the game is either 1 or 2.

3. When $q(1) = \bar{q}$, the continuation value is exactly equal to bid cost, so bidders are indifferent between bidding and not bidding at $p = 0$, and any $q(0) \in [0, 1]$ could be a part of an equilibrium strategy. The value to bidders is $v(0) = 0$, the probability of sale $1 - [1 - q(0)]^3 \in [0, 1]$, and the expected revenue (conditional on sale) is equal to $v = 3$. The ending price of the game is 0, 1, or 2, each with positive probability.

Suppose again that $v = 3 + \gamma$ for $\gamma \in (0, 1)$. Then, at $p = 1$, buyers are no longer indifferent between bidding and not bidding because, when a bidder manages to become a leader, the bidder gets $v^*(2) - 1 = 1 + \gamma$, which is strictly more than the bid cost. Therefore, in any SMPE, $q(1) = 1$, so $v^*(1) = 0$, and $v(1) = \frac{\gamma}{2}$. At $p = 0$, becoming a leader is not attractive because it is costly and gives 0 value; therefore, $q(0) = 0$. Even for arbitrarily small $\gamma > 0$, there is a unique SMPE, and in this equilibrium, the probability of sale is 0, and the value to players is 0.

Relation to related articles

The example above illustrates how my analysis differs from related literature, especially Augenblick (2012) and Platt, Price, and Tappen (2013). Their model is a special case that uses the single-bid assumption.
and assumes \( v - c = \lfloor v - c \rfloor \), which corresponds to the \( v = 3 \) case in the example above. Moreover, they focus on a particular equilibrium wherein, at each price \( p > 0 \), the non-leaders randomize in such a way that bidders at price level \( p - 1 \) are indifferent. The result is a unique price path after \( p > 0 \), and at \( p = 0 \), they assume that \( q(0) = 1 \).

As shown, the single-bid case with \( v = 3 \) was the case with a large number of SMPE with different implications for the outcomes of the game (realized prices and revenue). The equilibrium selected for analysis in [Augenblick (2012)] and [Platt, Price, and Tappen (2013)] is the one in which \( q(1) = \bar{q} \) and \( q(0) = 1 \) because it matches behavior in real penny auctions well and is tractable. However, as the above example shows, their selected equilibrium is not a unique SMPE, and it is not robust to small perturbations in the value of the object. In contrast, the simultaneous-bids case gives a unique equilibrium that is continuous in parameter values.

**General Analysis**

This section shows that most of the conclusions from the previous example hold generally. Although the simultaneous-bids assumption does not guarantee uniqueness, it has the following robust features: the probability of selling the object is always positive, there is always a positive probability that the ending price is high (above \( \lfloor v - c \rfloor \)), and the expected revenue is at most \( v \).

The equilibria under the single-bid assumption depend crucially whether \( v - c \) is an integer or not. When \( v - c \neq \lfloor v - c \rfloor \), there is a unique SMPE in which the object is either not sold or sold at price \( p = 1 \). When \( v - c \) is an integer, there is a continuum of equilibria, including the one chosen in [Augenblick (2012)] and [Platt, Price, and Tappen (2013)]. Regarding the type of equilibrium, the outcomes of the game include essentially the same features highlighted above, including positive probability that the realized price is above \( \lfloor v - c \rfloor \).

Remember that, according to lemma 4.1 in any game \( q(p) = 0, \forall p \geq \lfloor v - c \rfloor \). Define \( \gamma = v - c - \lfloor v - c \rfloor \) so that \( \gamma \in [0,1) \). When taking \( p = \lfloor v - c \rfloor + K = v - c - \gamma + K \) for \( K = 0,1, \ldots \), then \( q(p) = 0 \), and

\[
\begin{align*}
v^*(p) &= \max\{0,v - p\} = \max\{0,c + \gamma - K\}, \quad v(p) = 0. 
\end{align*}
\]

The rest of the game can be considered finite and solved using backwards induction. Take \( p < \lfloor v - c \rfloor \). If \( p > 0 \), there are \( N \) non-leaders, and if \( p = 0 \), there are \( N + 1 \). Denote the number of non-leaders by \( \bar{N} \). At each stage, the stage-game equilibrium could be one of three types: (1) all non-leaders submit bids with certainty, (2) all non-leaders choose not to bid, (3) all non-leaders randomize between bidding and not bidding. Next, I show how each of these three situations characterizes \( q(p), v^*(p) \), and \( v(p) \).

**Simultaneous-Bids Case**

Consider the three situations under the simultaneous bids assumption. First, there could be a stage-game equilibrium in which all \( \bar{N} \) non-leaders submit bids with certainty. In this case, \( q(p), v^*(p) \), and \( v(p) \) are characterized by the three equalities in conditions C1. This is an equilibrium if none of the non-leaders wants to pass and become non-leader at price \( p + \bar{N} - 1 \) with certainty, as characterized by the inequality in conditions C1.

\[\text{Expected revenue is equal to } v \text{ if an additional restriction is imposed on the equilibria.}\]
**Conditions 1** (C1). \(q(p) = 1, v^*(p) = v(p + 1), \) and

\[
v(p) = \frac{1}{N} v^*(p + N) + \frac{N-1}{N} v(p + N) - c \geq v(p + N - 1).
\]

Second, there could be a stage-game equilibrium in which all \(N\) non-leaders choose to pass. This is characterized in conditions C2.

**Conditions 2** (C2). \(q(p) = 0, v^*(p) = v - p,\) and \(v(p) = 0 \geq v^*(p + 1) - c.\)

Finally, there could be a symmetric mixed-strategy stage-game equilibrium in which all \(N\) non-leaders bid with probability \(q \in (0, 1).\) This is characterized in conditions C3.

**Conditions 3** (C3). \(0 < q(p) < 1,\)

\[
v(p) = \sum_{K=0}^{N-1} \binom{N-1}{K} q^K (1-q)^{N-1-K} \left[ \frac{1}{K+1} v^*(p + K + 1) + \frac{K}{K+1} v(p + K + 1) \right] - c
\]

\[
= \sum_{K=1}^{N-1} \binom{N-1}{K} q^K (1-q)^{N-1-K} v(p + K),
\]

\[
v^*(p) = (1-q)^N (v - p) + \sum_{K=1}^{N} \binom{N}{K} q^K (1-q)^{N-K} v(p + K).
\]

**Single bid case:** Under the single-bid assumption, the price always rises by 1, so the conditions in C1, C2, and C3 simplify, but the same three cases are still possible. The three sets of conditions take the following forms:

(C1) \(q(p) = 1, v^*(p) = v(p + 1), v(p) = \frac{1}{N}[v^*(p + 1) - c] + \frac{N-1}{N} v(p + 1),\) and \(v^*(p + 1) - c \geq v(p + 1).\)

(C2) \(q(p) = 0, v^*(p) = v - p, v(p) = 0,\) and \(v^*(p + 1) - c \leq 0.\)

(C3) \(0 < q(p) < 1,\)

\[
v^*(p) = (1-q)^N (v - p) + [1 - (1-q)^N] v(p + 1),
\]

\[
v(p) = v(p + 1) + \Psi_N(q)[v^*(p + 1) - c - v(p + 1)],
\]

\[
v^*(p + 1) - c = \left[ 1 - \frac{(1-q)^N}{\Psi_N(q)} \right] v(p + 1), \]

where \(\Psi_N(q) = \frac{1-(1-q)^N}{qN}.\) \(q \in [0, 1].\)

Note that in every equilibrium, each \(q(p)\) must satisfy either C1, C2, or C3, and therefore, an equilibrium is recursively characterized. However, nothing requires the equilibrium to be unique. Appendix C gives an example; in the example, at \(p = 2,\) each of the three sets of conditions gives different solutions, so there are three equilibria.

**Proposition 4.2.** When \(\varepsilon > 0,\) any SMPE \(q : \mathbb{N} \rightarrow [0, 1]\) with corresponding continuation value functions is recursively characterized by C1, C2, or C3 at each \(p < |v - c|\) and \(q(p) = 0\) for all \(p \geq |v - c|,\) where \([x]\) is the floor function. An equilibrium always exists but might not be unique.
Proof The construction above describes the method to find equilibrium \( q \). The conditions C1, C2, and C3 are written so that there are no profitable one-stage deviations. Proving the existence of SMPE only requires proof that there is at least one \( q \) that satisfies at least one of three sets of conditions.

Each round in the auction is a finite, symmetric strategic game. Theorem 2 in [Nash (1951)] proves that each symmetric strategic game has at least one symmetric equilibrium. Because the conditions are constructed so that any mixed- or pure-strategy–stage-game Nash equilibrium would satisfy them, there exists at least one such \( q \).

Appendix C gives a simple example in which the equilibrium is not unique.

Proposition 4.2 gives a general characterization of all equilibria in penny auctions with a positive bid increment. Although there are often many equilibria, as corollaries 4.3 and 4.4 below show, equilibria in penny auctions with a positive bid increment have some robust features and emphasize the differences between the simultaneous-bids assumption and the single-bid assumption.

First, corollary 4.3 shows that under the simultaneous-bids assumption, the outcomes of all SMPE are uncertain in the sense that the game may end at a very low price with positive probability as well as at a very high price with positive probability; the result is very high realized revenue for the seller. However, in all equilibria, the expected revenue is at most \( v \), and in equilibria in which the game may end in each period, the expected revenue is exactly \( v \).

Second, corollary 4.4 shows that the set of equilibria under the single-bid assumption follows a different pattern. Namely, when \( v - c \) is not an integer, the game ends very quickly (at price 0 or 1), and the revenue is very low. In contrast, when \( v - c \) is an integer, there is a large number of equilibria, which include not only very short equilibrium paths but also equilibria which have the characteristics highlighted above—high prices reached with positive probability and expected revenue \( v \).

Corollary 4.3. When \( \varepsilon > 0 \) and the simultaneous bids assumption holds, all SMPE have the following properties:

1. Expected revenue \( \overline{R} \leq v \). There exist parameter values wherein some equilibria \( \overline{R} < v \). If \( q(p) < 1 \) for all \( p \), then \( \overline{R} = v \).

2. If \( \gamma = (v - c) - |v - c| > 0 \), then \( q(|v - c| - 1) > 0 \). If \( \gamma = 0 \), then \( q(|v - c| - 1) > 0 \) or \( q(|v - c| - 2) > 0 \) or both. Price level \( p^* \geq v - c \) is reached with strictly positive probability.

3. The realized revenue \( R \geq v + (c + 1)(v - c - 1) \) with strictly positive probability.

Proof

1. The argument for \( \overline{R} \leq v \) is analogous to corollary 3.2. The aggregate expected value to the players must be equal to \( v \) minus the sum of payments to the seller; because participation is voluntary, the sum of payments cannot be larger than \( v \).

If \( q(p) < 1 \) for all \( p \), then this mixed strategy puts strictly positive probability on the pure strategy in which the player never bids. The pure strategy gives 0 with certainty, so \( v(0) = 0 \) for all players, and \( \overline{R} = v \) by implication.

If \( q(p) = 1 \) for some \( p \), then the previous argument does not work because the player does not put positive probability on the never-bidding, pure strategy. The example in appendix C gives an equilibrium (details are in table 2) in which \( q(0) \in (0, 1) \), \( q(1) = 0 \), but \( q(2) = 1 \), and \( E(R|p > 0) = 8.62 < 9.1 = v \).
2. Starting with the case $\gamma > 0$, by lemma [4.1] at $p = [v - c] = v - c - \gamma$, $q(p) = 0$; therefore, $v^*([v - c]) = c + \gamma$, and $v([v - c]) = 0$.

Take $p = [v - c] - 1$. Then C2 cannot be satisfied because $v^*(p + 1) - c = \gamma > 0$. Therefore, $q([v - c] - 1) > 0$. When $\gamma = 0$, there is no contradiction with C2 at $p = [v - c] - 1$, but there is one at $p = [v - c] - 2$ by the same argument.

Because $\bar{N}$ simultaneous bids raise the price by $\bar{N}$, the game stops by some price level $p^*$ only if there exist $\bar{N}$ sequential price levels before $p$, where $q(p)$. I claim that prices cannot be lower than $[v - c]$. Suppose, by contradiction, there exists $p < [v - c]$ such that $q(p) = q(p - 1) = 0$. Then $v^*(p) = v - p > v - [v - c] \geq c$. Then, at $p - 1$, $v^*(p - 1) - c > 0$, so C2 cannot be satisfied; consequently, there is a contradiction.

Therefore, when $\gamma = 0$, the game reaches $[v - c] - 2$ with positive probability, so the highest price reached with positive probability is $p^* \geq [v - c] - 2 + \bar{N} \geq v - c$. When $\gamma > 0$, $p^* \geq [v - c] - 1 + \bar{N} > v - c$.

3. As shown above, the final price is $p^* \geq v - c$ with positive probability, so revenue must be at least

$$R \geq (c + 1)(v - c) = v + (c + 1)(v - c - 1).$$

\[\square\]

**Corollary 4.4.** When $\varepsilon > 0$ and the single bid assumption holds, the set of equilibria depends on $\gamma = v - c - [v - c]$.

1. If $\gamma > 0$, the SMPE is unique. In particular, if $[v - c]$ is even, there are no bids, and the seller keeps the object. If $[v - c]$ is odd, all bidders submit bids in the first period and none afterward, so the revenue is 2, and the expected value to bidders is $\frac{v - 2}{N + 1}$.

2. If $\gamma = 0$, there is a continuum of equilibria including:

   (a) An equilibrium in which all players are always indifferent\[10\] for all $p < [v - c]$, the probability is chosen so that $v(p) = 0 = v^*(p) - c$ and condition C3 is satisfied. In fact, there is a continuum of such equilibria because $q(0)$ could be arbitrary. In this class of equilibria, conditional on sale, each price $\{1, \ldots, [v - c]\}$ occurs with strictly positive probability.

   (b) An equilibrium that is robust to small changes in $\gamma$. In this equilibrium, the game ends either after 0 or 1 bid depending on whether $v - c$ is even or odd.

**Proof**

1. If $\gamma > 0$, solve backwards:

   0) At $p = [v - c] = v - c - \gamma$, so $v^*(p) = v - p = c + \gamma$, and $v(p) = 0$.
   1) At $p = [v - c] - 1$, $v^*(p + 1) - c = \gamma > 0 = v(p + 1)$, so only C1 is satisfied; thus, $q(p) = 1, v^*(p) = 0, v(p) = \frac{\gamma}{N}$.
   2) At $p = [v - c] - 2$, $v^*(p + 1) = 0 < v(p + 1)$, so only C2 is satisfied; thus, $q(p) = 0$, and $v^*(p) = v - p = c + \gamma + 2$ and $v(p) = 0$.

\[10\]As pointed out in the discussion above, this was the equilibrium picked by Augenblick (2012) and Platt, Price, and Tappen (2013).
K) Repeat the same pattern: At each \( p = \lfloor v - c \rfloor - K \) with odd \( K \), only C1 can be satisfied, and with even \( K \), only C2 can be satisfied.

Therefore, at \( p = 0 \), C1 is satisfied if \( \lfloor v - c \rfloor \) is odd, and C2 is satisfied if it is even, so the SMPE is unique.

2. If \( \gamma = 0 \), there is a continuum of equilibria including:

(a) Proof by induction:

0) At \( p = \lfloor v - c \rfloor = v - c \), \( v^*(p) = v - p = c \), and \( v(p) = 0 \).
1) At \( p = v - c - 1 \), \( v^*(p + 1) - c = v(p + 1) = 0 \), so C3 is satisfied for any \( q \). Therefore, \( q \) can be chosen such that \( v^*(p) = (1 - q)^N(c + 1) = c \); that is, \( q(p) = \left[ \frac{c}{c+1} \right]^N \), and \( v(p) = v(p+1) = 0 \).

K) At any \( p = \lfloor v - c \rfloor - K \), \( v^*(p+1) - c = v(p+1) = 0 \), so C3 is satisfied for any \( q \). Therefore, \( q \) can always be chosen such that \( v(p) = (1 - q)^N(c + K) = c \), and \( v(p) = 0 \).

This construction determines a unique sequence \( q(1), q(2), \ldots \), but not \( q(0) \), which can be arbitrary. Because each \( q(p) \) is interior, the probability that the game ends at any \( p \in \{1, \ldots, \lfloor v - c \rfloor \} \) is strictly positive.

(b) The construction is the same as in part 1 of the proof, but the inequalities are weak when \( \gamma = 0 \), so the equilibrium is not unique.

\[ \square \]

5 Conclusions

In this article, I have studied penny auctions, a novel auction format that leads to unpredictable outcomes in practice. I have proposed a tractable model of penny auctions and showed that, in some sense, unpredictability is a property of the auction format. In particular, under the simultaneous-bids assumption, all SMPE must be such that very high prices and very high revenue occurs with strictly positive probability.

I have also characterized SMPE under the single-bid assumption, which is a standard assumption in most of the related literature, starting with Augenblick (2012). In fixed-price penny auctions, results are similar to those modeled with the simultaneous-bids assumption, but there are two differences: (1) bidding probability at the initial stage is not determined, and (2) aggregate behavior is independent of the number of bidders (and this independence is a testable implication). In increasing-price penny auctions, the set of equilibria under the single-bid assumption depends critically on whether a particular parameter of the game, \( \frac{V - C}{\varepsilon} \), is an integer (as assumed by other authors) or not. If it is not an integer, there is a unique equilibrium, and the game ends very quickly. When it is an integer, there is a continuum of equilibria, including the one other authors focus on. The equilibrium that other authors focus on has properties similar to the equilibrium in the simultaneous-bids case, but it is not robust to small changes in parameter values. The only equilibria that are robust to changes in parameter values are ones in which the game ends very quickly.

Although the equilibria match the bidding pattern and end prices quite well, the model is unable to explain why real penny auctions have average profit margins higher than zero. Under the risk-neutrality and rationality assumptions, the seller’s revenue cannot be higher than the value of the object. Because individuals can always ensure at least zero value by inactivity, it is impossible to extract on average
more than the value they expect to get. To achieve an outcome in which the expected revenue is strictly higher than the value of the object, the model needs to be expanded. Potential explanations are intrinsic value from participating or winning\(^{11}\) and risk-loving preferences. It is also possible that the bidders in penny auctions are not fully rational so that high revenue could be explained by the sunk-cost bias, prospect-theory preferences, or simply mistakes due to inexperience.

Finally, the results in this article and in the related literature point to an interesting area of future research: perhaps, these auctions are good for raising money for public goods. \cite{Goeree2005} shows that it is better to raise money for public goods with all-pay auctions than with winner-pay auctions. \cite{Carpenter2014} finds experimentally that their physical implementation of penny auctions (which they call *bucket auctions*) raised even more money than four alternative all-pay auction formats, and they attribute the difference to sunk-cost sensitivity.

References


\(^{11}\)After all, penny auctions are typically advertised as “entertainment shopping.”
Appendix A  Stylized Facts about Penny Auctions

The data used in this appendix comes from Swoopo, the largest penny auction site at the beginning of May 2009 when data were collected. Data from about 61,153 auctions were collected directly from their website; only auctions with complete data were included. Each auction had information about the auction type, the value of the object (suggested retail price), delivery cost, the winner’s identity, and the number of free and costly bids the winner made (which was used to calculate “the savings”). The number of observations and some statistics to compare the orders of magnitude are given in table 1. The main Swoopo auction types that are relevant for the current article were the following (all four are special cases of the auction format discussed in this paper, in all cases the bid cost is $0.75):

1. A **penny auction** is an auction with a price increment of $0.01.
2. A **regular auction** is an auction with a price increment of $0.15.
3. In a **fixed-price auction**, the winner pays the pre-announced, fixed price instead of the ending price of the auction.
4. A **free auction** (or **100% off auction**) is a special case of a fixed-price auction in which the winner pays only the delivery charges.

![Insert table 1 here]

Figure 1 describes the distribution of final prices in different auction formats. To compare prices of objects with different values, the plot is normalized by the value of the object. The final price equals the retail price at 100%. Most auction formats have similar distributions with relatively high mass at low values and long tails. It is not surprising that penny auctions are much more concentrated on low values because the bidders have to make 15 times more bids than in other formats to reach any particular price level.

![Insert figure 1 here]

The most intriguing fact in figure 1 should be the positive mass in relatively high prices because the cumulative bid costs to reach high prices can be much higher than the value of the object. The wide range of ending prices implies that the profit margins to the seller and winner’s payoff are very volatile. Indeed, figure 2(a) describes the distribution of the profit margin and there is positive mass in very high profit margins. The figure is somewhat arbitrarily truncated at 1000%; there is also positive, but small, mass at much higher margins.

![Insert figure 2 here]

---

12 In March 2011, Swoopo went bankrupt, and the website is not accessible anymore.
13 Profit margin is simply defined as $\frac{\text{End price} + \text{Total bid costs} - \text{Value}}{\text{Value}} \times 100$.
14 In the dataset, 7.12% of the bids were free I used 92.88% of $0.75, which is $0.6966, as the bid cost.
Similarly, figure 2(b) describes the winner’s savings\footnote{Swoopo.com defines the winner’s savings as the difference between the value of the object and winner’s total cost divided by the value. Note that the reported savings at the website are such that the negative numbers are replaced by 0.} from different types of auctions. In this plot, zero on the x-axis means no savings compared to retail price, so to the left of this value, even the winner would have gained by just purchasing the object from a retail store. In most cases, the winner’s savings are highly positive, and winner’s savings probably explains why agents participate in the auctions after all. The density of the winnings is increasing in all auction formats with the mode near 100%, but the auction formats differ. Regular auctions have a relatively low mean and the flattest distribution, whereas free auctions have highest mean and more mass concentrated near 100%. These results fit expectations because, in these auctions, the cost is relatively more equally distributed between the bidders. (If the winners were making most of the bids, they would win very early.)

[Insert figure 3 here]

Finally, figure 3 shows the distribution of the total number of bids. The frequencies slowly decrease as the number of bids increases, except in low numbers of bids where the decrease is rapid. In penny auctions there are, on average, many more bids than in other formats.

Appendix B  
Symmetric Markov-Perfect Equilibria

In this appendix, I formally introduce the equilibrium concept used in this article, SMPE, and prove proposition 2.1. First, I introduce notation for general strategies to formally define the equilibrium concept and to describe deviations in the equilibrium characterization.

Let the vector of bids at period $t$ be denoted by $b_t = (b_0^t, \ldots, b_N^t)$, where $b_i^t = 1$ if player $i$ submitted a bid at period $t$ and 0 otherwise. Let the leader after $t$ period be $l_t \in \{0, \ldots, N\}$. The information that each player has when making a choice at period $t$, or history at $t$, is $h_t = (b_0^0, l_0^1, b_1^1, l_1^1, \ldots, b_{t-1}^{t-1}, l_{t-1}^{t-1})$.

The game sets some restrictions on the possible histories. In particular, to become a leader, one must submit a bid; therefore, $b_l^t = 1$; the leader cannot submit a bid, $b_{l_t}^{t-1} = 0$; and $h_t$ is defined only if none of the previous bid vectors $b^r$ is zero vector. Denote the set of all possible $t$-stage histories by $H^t$ and the set of all possible histories by $H = \bigcup_{t=0}^\infty H^t$.

In this game, a pure strategy of player $i$ is $b_i : H \rightarrow \{0, 1\}$, where $b_i(h_t) = 1$ means the player submits a bid at $h_t$ and $b_i(h_t) = 0$ means that the player passes. The strategies\footnote{For the winners, the number of free bids is known, so this is taken into account precisely.} of the players are $\sigma_i : H \rightarrow \{0, 1\}$ such that $\sigma_i(h_t)$ is the probability that player $i$ submits a bid at history $h_t$. Note that by the rules of the game, $\sigma_i(h_t) = 0$ at all histories $h_t$, where $l_t = i$. (The leader can only pass.)

A strategy profile $\sigma$ is symmetric if for all $t \in \{0, 1, \ldots\}$, for all $i, \hat{i} \in \{0, \ldots, N\}$, and for all $h_t = (b^r, l^\tau)_{\tau=0,\ldots,t-1} \in H^t$; if $\hat{h}_t = (\hat{b}^r, \hat{l}^\tau)_{\tau=0,\ldots,t-1} \in H^t$ satisfies

\[
\begin{align*}
\hat{b}^r_j &= \begin{cases} 
 b_j^r & \forall j \notin \{i, \hat{i}\}, \\
 b_j^r & j = \hat{i}, \\
 b_j^r & j = i,
\end{cases} \\
\hat{l}^\tau &= \begin{cases}
 l^\tau & \forall \tau \notin \{i, \hat{i}\}, \\
 i & l^\tau = \hat{i}, \\
 \hat{i} & l^\tau = i,
\end{cases} \quad \forall \tau = \{0, \ldots, t-1\},
\end{align*}
\]

then $\sigma_i(\hat{h}_t) = \sigma_i(h_t)$. Informally, the symmetry assumption simply states that when two players switch identities, nothing changes.

\footnote{That is, the leader is the non-leader that submitted a bid at $t$ and became the leader by random draw.}

\footnote{The game has perfect recall, so by Kuhn’s theorem, any mixed strategy profile can be replaced by an equivalent behavioral strategy profile. For simpler notation, strategies in the text means behavioral strategies.}
Let function \( L_i \) be the indicator function that tells whether player \( i \) is leader after history \( h^t \), where \( L_i(h^t) = 1[i = l^t] \). Let \( S \) be the set of states in the game, and let \( S : \mathcal{H} \to S \) be the function mapping histories to states. More specifically,

(i) If \( \varepsilon = 0 \), then \( S = \{N + 1, N\} \), and \( S(h^t) = N + 1 \) if \( t = 0 \) and \( N \) otherwise. Because the price does not increase, the only payoff-relevant characteristic for the players is the number of active bidders, which is \( N + 1 \) in the beginning and \( N \) at any round after 0.

(ii) If \( \varepsilon > 0 \), then \( S = \mathbb{Z}_+ \), and \( S(h^t) = \sum_{t=0}^{t-1} \sum_{i=0}^{b_i^t} \); that is, the total number of bids made so far or, equivalently, the normalized price. Note that it is unnecessary to explicitly consider two cases with two different numbers of players because at \( h^t = \emptyset \) \( S(h^t) = 0 \) and at any other history \( S(h^t) > 0 \).

A strategy profile \( \sigma \) is Markovian if for all pairs of histories \( h^t = (b^t, l^t)_{\tau=0}^{t-1} \in \mathcal{H}, \ \hat{h}^t = (\hat{b}^t, \hat{l}^t)_{\tau=0}^{t-1} \in \mathcal{H} \) such that \( L_i(h^t) = L_i(\hat{h}^t) \) and \( S(h^t) = S(\hat{h}^t) \), so \( \sigma_i(h^t) = \sigma_i(\hat{h}^t) \). That is, if two histories lead to the same state and under both histories player \( i \) is not leader, then the probability of bidding in both situations must be the same. (If player \( i \) is the leader, the claim is trivially satisfied.)

A sub-game perfect Nash Equilibrium (SPNE) strategy profile \( \sigma \) is SMPE if \( \sigma \) is symmetric and Markovian.

**Proposition B.1** (Proposition 2.1). A strategy profile is SMPE if and only if it can be represented by \( q \) where \( q(\cdot) \) is the Nash equilibrium of the stage-game in a particular state, taking into account the continuation values.

I split the proof into three lemmas. First, lemma B.2 proves the representation part; that is, any SMPE can be represented by \( q(\cdot) \), and vice versa, if a SPNE equilibrium can be represented by \( q(\cdot) \), then it is an SMPE. Lemmas B.3 and B.4 prove the characterization results for \( \varepsilon > 0 \) and \( \varepsilon = 0 \), respectively. In both cases, I show that the game can be essentially solved by looking at one-shot deviations. Note that, because the game is without discounting and (in both cases) potentially infinite, continuity at infinity is not satisfied, so it is not valid to simply apply the one-shot deviation principle. The results will prove that it is still possible to use standard characterization techniques.

**Lemma B.2.** A strategy profile \( \sigma \) is symmetric and Markovian if and only if it can be represented by \( q : S \to [0, 1] \), where \( q(s) \) is the probability bidder \( i \) bids at state \( s \in S \) for each non-leader \( i \in \{0, \ldots, N\} \).

**Proof** Because \( q \) is only defined on states \( S \) and equally for all non-leaders, it is obvious that it is symmetric and Markovian; therefore, sufficiency is trivially satisfied. For necessity, take any symmetric Markovian strategy profile \( \sigma = (\sigma_0, \ldots, \sigma_N) \), where \( \sigma_i : \mathcal{H} \to [0, 1] \). Construct functions \( q_0, \ldots, q_N \), where \( q_i : S \to [0, 1] \), by setting

\[
q_i(S(h^t)) = \begin{cases} 
0 & \forall h^t : L_i(h^t) = 1, \\
\sigma_i(h^t) & \forall h^t : L_i(h^t) = 0, 
\end{cases} \quad \forall h^t \in \mathcal{H}.
\]

The construction of \( S \) and the Markovian property ensure that \( q_i \) is a well-defined function.

I claim that adding symmetry means that \( q_i(s) = q(s) \) for all \( i \) and \( s \in S \). Fix any \( i \) and \( h^t \) such that \( s = S(h^t) \) and \( L_i(h^t) = 0 \). By construction, \( q_i(s) = q_i(S(h^t)) = \sigma_i(h^t) \). Next, fix any other non-leader, \( \hat{i} \), so that \( L_{\hat{i}}(h^t) = 0 \). Construct another history \( \hat{h}^t \) that is otherwise identical to \( h^t \) except that \( i \) and \( \hat{i} \) are switched. Then, \( S(\hat{h}^t) = s \) (which is obvious for both cases), and \( L_{\hat{i}}(\hat{h}^t) = 0 \). By symmetry, \( \sigma_{\hat{i}}(h^t) = \sigma_{\hat{i}}(\hat{h}^t) \). Therefore, \( q_i(s) = q_i(S(\hat{h}^t)) = \sigma_{\hat{i}}(\hat{h}^t) = \sigma_{\hat{i}}(h^t) = q_i(s) \). \( \square \)
This result means that if a strategy profile is symmetric and Markovian, it can be simplified. Replace \( \sigma \) by \( q \) which is just defined for all \( s \in S \) instead of the full set of histories \( H \). The following two lemmas show that (at least in the cases considered in this article) the solution method is also simplified by these assumptions because any SMPE can be found simply by solving for stage-game Nash equilibria for each state \( s \in S \), taking into account the solutions to other states and the implied continuation value functions.

**Lemma B.3.** With \( \varepsilon > 0 \), a strategy profile \( \sigma \) is a SMPE if and only if it can be represented by \( q : S \to [0,1] \) where \( q(s) \) is the Nash equilibrium in the stage-game at state \( s \), taking into account the continuation values implied by transitions \( S \).

**Proof**

(i) Necessity: If \( \sigma \) is SMPE, then by lemma B.2, it can be represented by \( q \); additionally, because it is a SPNE, there cannot be profitable one-stage deviations.

(ii) Sufficiency: When \( \varepsilon > 0 \), the price rises each period if the game has not stopped. Because rational buyers will never bid after the price rises above the value of the object, the game stops after a finite number of periods. So, although the game is (by the rules) infinite, it is equivalent in the sense of payoffs and equilibria with a game which is otherwise identical to the initial auction but where, after some finite time, the current leader gets the object at the current price. This game is finite, and checking one-stage deviations is a sufficient condition for SPNE.

**Lemma B.4.** With \( \varepsilon = 0 \), a strategy profile \( \sigma \) is a SMPE if and only if it can be represented by \( q : S \to [0,1] \) where \( q(s) \) is the Nash equilibrium in the stage-game at state \( s \), taking into account the continuation values implied by transitions \( S \).

**Proof**

Sufficiency: Suppose \( q \) is a Nash equilibrium in the stage-game at each state \( s \). Let us shorten the notation as follows: The probability that a non-leader submits a bid at state \( N \) is \( \hat{q} \), and the expected value for a non-leader is \( \hat{v} \); for the leader, the expected value is \( \hat{v}^* \). Similarly, at state \( N + 1 \) (i.e., at period 0), the corresponding values are \( \hat{q}_0, \hat{v}_0, \) and \( \hat{v}^*_0 \), respectively. Finally, let the probability that the buyer who submitted a bid becomes a leader be \( P \), which depends on \( \hat{q} \) because it depends on what the opponents do but not on buyer’s own strategy.

Consider any period after period 0. By assumption, in equilibrium, non-leaders should submit bids with probability \( \hat{q} \). Clearly, \( \hat{q} \in (0,1) \) so that each of them is indifferent between bidding and not bidding. That is,

\[
P\hat{v}^* + (1-P)\hat{v} - Z(P)c = \hat{v} \iff \hat{v}^* - \hat{v} = \frac{Z(P)}{P}c,
\]

where \( Z(P) = 1 \) in the simultaneous-bids case (in which each bid costs) and \( Z(P) = P \) in the single-bid case (in which only the new leader pays). Moreover, the leader’s value must satisfy \( \hat{v}^* = (1-\hat{q})^NV + [1 - (1-\hat{q})^N]\hat{v} \) so that \( \hat{v}^* - \hat{v} = (1-\hat{q})^N[v - \hat{v}] \).

Suppose that a non-leader has a profitable multi-stage deviation at period \( t \) (still assuming that opponents use strategy \( \hat{q} \)). Suppose at \( t \), this strategy requires submitting a bid at point \( t \) with probability \( Q \) and assures value \( V \) at period \( t \). Then, at any point in the future when the player is a non-leader...
again, the value must be \( V \) again because, otherwise, the leader could increase the value at either point. Also, let \( V^* \) be the value for the bidder at each point in the future when the bidder is a leader. (\( V^* \) must be equal in all points for the same reason.) Then,

\[
V = Q\left[ PV^* + (1 - P)V - Z(P)c \right] + (1 - Q)V \implies V^* - V = \frac{Z(P)}{P}c.
\]

The game ends with probability \((1 - \hat{q})^N\). Therefore, \( V^* = (1 - \hat{q})^N v + [1 - (1 - \hat{q})^N]V \) so that \( V^* - V = (1 - \hat{q})^N [v - V] \). As a result,

\[
(1 - \hat{q})^N [v - V] = \frac{Z(P)}{P} = (1 - \hat{q})^N [v - \hat{v}] \implies V = \hat{v}.
\]

This result contradicts the assumption that \( V > \hat{v} \).

I have shown that always playing \( \hat{q} \) at any period after 0 ensures highest possible payoffs. Therefore, if there is profitable deviation at period 0, it must be one-stage deviation. But profitable deviation is not possible because it contradicts the assumption that \( \hat{q}_0 \) is a Nash equilibrium in the stage-game, taking into account the continuation values from \( \hat{q} \) in the following periods.

\[\square\]

**Appendix C  Penny Auction with Multiple Equilibria**

Let \( N + 1 = 3, v = 9.1, c = 2, \) and \( \varepsilon > 0, \) and suppose that the simultaneous-bids assumption holds. There are three SMPE, which are described in tables 2 to 4.

[Insert tables 2 to 4 here.]
Figure 1: Distribution of the normalized end prices in different types of auctions

Figure 2: Distribution of the profit margin and winner’s savings in Regular and Free auctions.

Figure 3: Distribution of the number of bids submitted in different types of auctions
Table 1: General descriptive statistics about the auctions. The symbols $v$ and $c$ refer to normalized variables introduced in the next section. The average number of bids can be approximately interpreted as the normalized price, $p$.

<table>
<thead>
<tr>
<th>Type of auction</th>
<th>Observations</th>
<th>Average value ($)</th>
<th>Average price ($)</th>
<th>Norm. value, $v$</th>
<th>Norm. cost, $c$</th>
<th>Avg. # of bids</th>
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<td>Regular</td>
<td>41760</td>
<td>166.9</td>
<td>46.7</td>
<td>1044</td>
<td>5</td>
<td>242.9</td>
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<tr>
<td>Penny</td>
<td>7355</td>
<td>773.3</td>
<td>25.1</td>
<td>75919.2</td>
<td>75</td>
<td>1098.1</td>
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<tr>
<td>Fixed price</td>
<td>1634</td>
<td>967</td>
<td>64.9</td>
<td>6290.7</td>
<td>5</td>
<td>2007.2</td>
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<td>Free</td>
<td>3295</td>
<td>184.5</td>
<td>0</td>
<td>1222</td>
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<td>558.5</td>
</tr>
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<td>All auctions</td>
<td>61153</td>
<td>267.6</td>
<td>41.4</td>
<td>10236.3</td>
<td>13.4</td>
<td>420.9</td>
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Table 2: Equilibrium with $q(2) = 1$

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<th>$v^*(p)$</th>
<th>$v(p)$</th>
<th>$Q(p)$</th>
<th>$Q(p)$</th>
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<td>0.3681</td>
<td>0.4175</td>
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<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>0.6996</td>
<td>0.5504</td>
<td>0</td>
<td>0.0119</td>
<td>0.0135</td>
</tr>
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<td>5.1</td>
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<td>0.4958</td>
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<tr>
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<td>0.0277</td>
<td>0.0314</td>
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<td>0.0178</td>
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</table>

Table 3: Equilibrium with $q(2) = 0.7249 \in (0, 1)$

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<th>$v^*(p)$</th>
<th>$v(p)$</th>
<th>$Q(p)$</th>
<th>$Q(p)$</th>
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Table 4: Equilibrium with $q(2) = 0$

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<th>$Q(p)$</th>
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