Penny Auctions

Toomas Hinnosaar*

May 22, 2009

Abstract

This paper studies a new form of auctions called penny auctions. In these auctions, strange things happen. Sometimes an auction selling cash generates revenue an order of magnitude higher than amount of cash sold, sometimes an order of magnitude lower. Sometimes the winner of the auction pays an order of magnitude less than many losers losers at the same auction. We will describe stylized facts about these auctions in practice and propose a tractable model that replicates these facts. We show that in this model, the actual outcomes and the revenue for the seller are in general be very volatile and characterize conditions under which the expected revenue takes its maximal value.

1 Introduction

A typical penny auction[1] may sell a new brand-name digital camera, at starting price 0 and timer at 1 minute. When the auction starts, the timer starts to tick down and players may submit bids. Each bid costs \$1 to the bidder, increases price by \$0.01, and resets the timer to 1 minute. Once the timer ticks to 0, the bidder who made the last bid can purchase the object at the current price. Note that the structure of penny auctions is similar to dynamic English auctions, but with just one significant difference. In penny auction the bidder has to pay significant price for each bid she makes.

Both the name and general idea of the auction is very similar to the dollar auction introduced by Shubik (1971). In this auction cash is sold to the highest bidder, but the two highest bidders will pay their bids. Shubik used it to illustrate potential weaknesses of traditional solution concepts and described

*t-hinnosaar@northwestern.edu.

[1]There are two kinds of practical auctions where the name penny auctions has been used. First type was observed during the Great Depression, foreclosed farms were sold at the auctions. In these auctions sometimes the farmers colluded to keep the farm in the community at marginal prices. These low sales prices motivated the name penny auctions. Second use of the term comes from the Internet age, where in the auction sites auctions are sometimes started at very low starting prices to generate interest in the auctions. Both uses of the term are unrelated to the auctions analyzed in this paper.

this auction as extremely simple, highly amusing, and usually highly profitable for the seller.

Dollar auction is a version of all-pay auction, that has used to describe rent-seeking, R&D races, political contests, and job-promotions. Full characterization of equilibria under full information in one-shot (first-price) all-pay auctions is given by Baye, Kovenock, and de Vries (1996). A second-price all-pay auction, also called war of attrition, was introduced by Smith (1974) and has been used to study evolutionary stability of conflicts, price wars, bargaining, and patent competition. Full characterization of equilibria under full information is given by Hendricks, Weiss, and Wilson (1988). Siegel (2009) provides a equilibrium payoff characterization for general class of all-pay contests.

Penny auction is clearly an all-pay auction, but not a special case of well known auctions mentioned above. None of the auctions mentioned above allow the actual winner to pay less than the losers, but in penny auctions it happens in practice relatively often. Finally, in a special case where bid cost converges zero, penny auction is converging to a dynamic first price auction. In this paper we only consider auctions with strictly positive bid costs. We will show that in the limit where bid costs are close to zero, the object is never sold.

Section 2 describes how penny auctions are used in practice and shows some stylized facts. Section 3 introduces the model and assumptions. The analysis is divided into two parts. Section 4 analyzes the case when the price increment, or “the penny” in the auction name is zero, which means that the auction game will be infinite. Section 5 discusses the case, where price increment is strictly positive. Section 6 gives some concluding remarks and suggests extensions for the future research.

2 Stylized facts

The data used in this section comes from Swoopo.com[2], a large penny auctions site. The data about 61,153 auctions were collected directly from their website and includes all auctions that had complete data in the beginning of May 2009[3] Each auction had information about the auction type, the value of the object (suggested retail price), delivery cost, the winners identity and the number of free and costly bids the winner made (used to calculate “the savings”), and the identities of 10 last bidders with information whether the bid was made using BidButler[4] or not (594,956 observations in total).

All auctions in Swoopo have the same structure as described in this paper, but they have several different types of auctions which imply different parameter values. Their main auction types are the following[5].

[2]See `http://www.swoopo.com/what_is.html` for details.

[3]Auctions that had incomplete data or had not finished were excluded from the dataset.

[4]BidButler is an automatic bidding system where user fixes minimum and maximum price and the number of bids between them and the system makes bids for them according to some semi-public algorithm.

[5]Auctions also differ by the length of timer, ie in 20-Second Auction if after the last submitted bid the timer ticks 20 seconds, the auction ends.

- Regular auction[6] is a penny auction with price increment of \$0.15 and bid cost of \$0.75[7].
- Penny auction is an auction where price increment is \$0.01 instead of \$0.15.
- Fixed Price Auction, where at the end of auction the winner pays some pre-announced fixed price instead of the ending price of the auction. The Free Auction (or 100% Off Auction) is a special case of Fixed Price Auction where the winner pays only the delivery charges.[8] Both of these auction have the property that price increment is zero, which means that there is no clear ending point and the auctions could in principle continue infinitely.
- NailBiter Auction is an auction where BidButlers are not allowed, so that each bid is made by actual person clicking on the bid button.
- Finally there are some variations regarding restrictions about customers who can participate. If not specified otherwise, everyone who has won less than eight auctions per current calendar month can participate. Beginner Auction is restricted to customers who have never won an auction. Open Auction is an auction where the eight auction limit does not apply, so it is fully unrestricted.

The number of observations and some statistics to compare the orders of magnitude are given in the Table 1. Column $\overline{V}$ is the average retail value and $\overline{P}$ the average price paid by the winner. Variables v, c, and p are normalized value, bid cost, and end price that we will discuss and use later. Note that p can be interpreted as the average number of bids made in the auction. As we can see, objects sold in different types are quite different and induce differing outcomes.

Figure 1 describes the distribution of end prices in different auction formats. To be able to compare the prices of objects with different values, the plot is normalized by the value of object. For example 100 means that final price equals the retail price. Clearly penny auctions have different distribution as expected, since to reach any particular price level, in penny auction the bidders have to make 15 times more bids than in other formats. From the other formats, it seems that the nailbiter and beginner auctions have somewhat bigger mass at low values, but otherwise the distributions are relatively similar.

The most intriguing fact in the Figure 1 should be the positive mass in relatively high prices, since the cumulative bid costs to reach to these prices can

[6] In the calculations below, we call the auction regular if it is not any of the other types of the auctions, but the other types are not mutually exclusive. For example auction can be a nailbiter penny auction with fixed price, so it is included in calculations to all three types.

[7] In all auction formats, \$0.75 is the standard price, which is actually the upper bound of bid cost, since bids can be purchased in packages so that they are cheaper and perhaps also sunk. Also, sometimes bids can be purchased at Swoopo auction at uncertain costs.

[8] Note that both Fixed Price Auctions and Free Auctions were discontionued by 2009, "in order to provide a more convenient bidding experience".

Type	Obs	$\overline{V}$	$\overline{P}$	v	c	p (# of bids)
Regular	41760	166.9	46.7	1044	5	242.9
Penny	7355	773.3	25.1	75919.2	75	1098.1
Fixed price	1634	967	64.9	6290.7	5	2007.2
Free	3295	184.5	0	1222	5	558.5
Nailbiter	924	211.5	8.3	1394.1	5	580.1
Beginner	6185	214.5	45.8	1358.5	5	301.6
All auctions	61153	267.6	41.4	10236.3	13.4	420.9

Table 1: Some statistics about the auctions

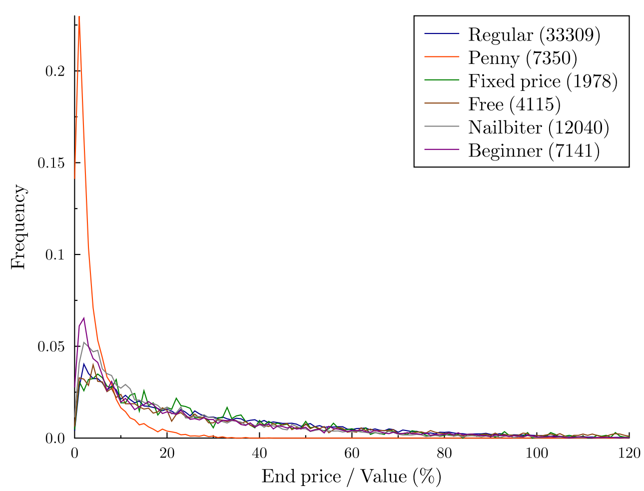

Figure 1: Distribution of normalized end prices in different types of auctions

be much higher than the value of the object. This implies that the profit margins to the seller and winner's payoff are very volatile. Indeed, Figure 2 describes the distribution of the profit margin[9][10] in different auctions, and there is positive mass in very high profit margins. The figure is somewhat arbitrarily truncated at 1400%, there also is positive, but small mass at much higher margins. Note that the profit margin is calculated relative to suggested retail value, so that zero profit margin should be sufficient profit for a retail company, but mean profit margin is positive for all the auctions.

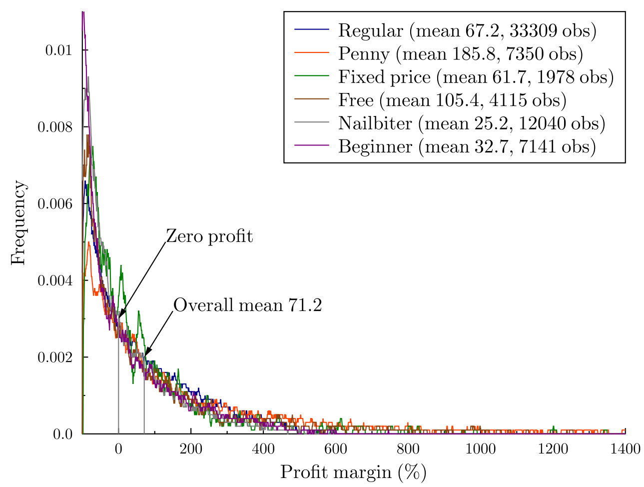

Figure 2: Distribution of the profit margin in different types of auctions

Similarly, Figure 3 describes the winner's savings[11][12] from different types of auctions. In this plot, 0 would mean no savings compared to retail price, so that on the left of this line even the winner would have gained just by purchasing the object from a retail store. Mostly the winner's savings are highly positive, which is probably the reason why agents participate in the auctions after all.

[9] Profit margin is simply defined as $\frac{\text{End price}+\text{Total bid costs}-\text{Value}}{\text{Value}} \cdot 100$.

[10] To make the plot, we need an approximate for the average bid costs. Official value is $0.75, but it is possible to get some discounts and free bids, so this would be the upper bound. In the dataset we have the number of free and non-free bids that the winners made and it turns out that about 92.88% of the bids are not free, so we used 92.88% of $0.75 which is $0.6966 as the bid cost. The overall average profit margin would be 0 at average bid cost $0.345, which is about two times smaller than our approximation of the average bid cost.

[11] Defined by Swoopo.com as the difference between the value of the object and winner's total cost divided by the value. Obviously, the losers will not save anything and the winner cannot ensure winning, so the term "savings" can be misleading in ex-ante sense.

[12] Again, the question is what is the right average bid cost to use. For the winners we know the number of free bids, so this is taken into account precisely, but for the costly bids, the we used the official value $0.75. True value may be below it, since there could be some quantity discounts, but it does not take into account any other constraints (like cost of time and effort). However, winner's average savings are positive for bid costs up to $2.485, which is far above the reasonable upper bounds of the bid cost.

The density of the winnings is increasing in all auctions with mode near 100%, but the auctions differ. The regular auctions have lowest mean and flattest distribution, whereas penny auctions, and the two fixed price auction types have highest mean and more mass concentrated near 100%. This is what we would expect, since in penny auction the actual price rises slowly and in fixed price auctions it does not change at all, so that the cost is relatively more equally distributed between the bidders (if the winner was the one making most of the bids, she would win very early).

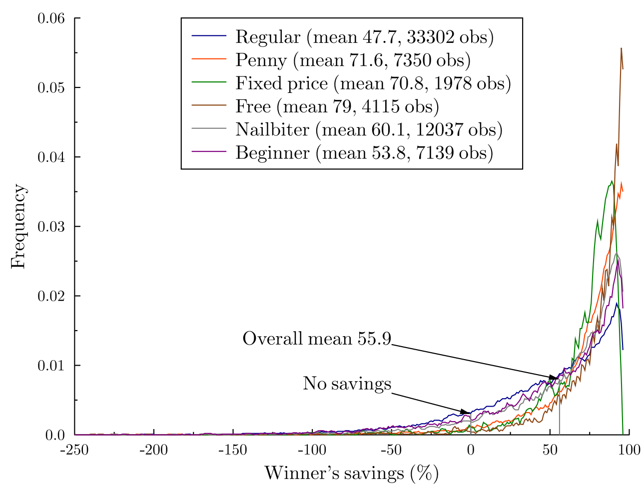

Figure 3: Distribution of the winner's savings in different types of auctions

The final piece of stylized facts we are looking here is the distribution of the number of bids. Figure 4 shows the distribution of the number of bids made in most auctions. In this figure we can see that two types of auctions are very different than other — in fixed price auctions[13] and penny auctions the auction generally lasts much longer than in other types. The type where auction ends at relatively low number of bids relatively more often is the nailbiter auction, where the bidders cannot use automated bidding system.

3 The Model

The goal of the next three sections is to introduce a framework that generates the stylized facts from the previous section. We will discuss some extensions in Sections 6.

The auction sells an object with market price of V dollars. This is fixed and common value to all the participants. There are $N+1 \geq 2$ players (bidders) participating in the auction, denoted by $i \in \{0, 1, \ldots, N\}$. We assume that

[13]The fact that Free auctions and Fixed Price auctions look different in this figure is somewhat surprising and explaining this would probably require more careful empirical analysis.

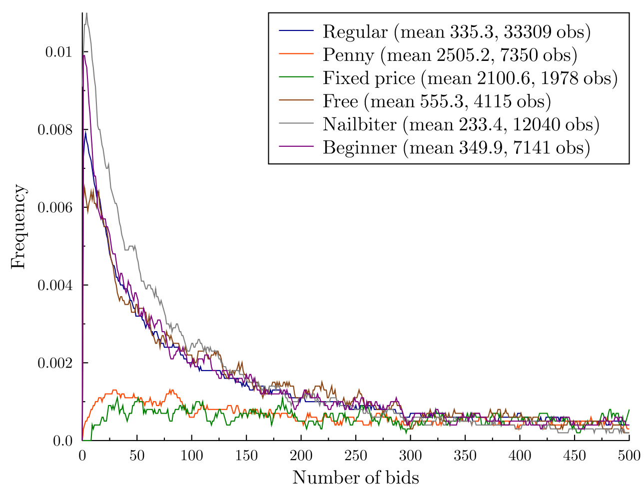

Figure 4: Distribution of the number of bids submitted in different types of auctions, range of low values

all bidders are risk-neutral and at each point of time maximize the expected continuation value of the game (in dollars).

The auction is dynamic, bids are submitted in discrete time points $t = 0, 1, \ldots$. Auction starts at initial price P_0[14]. At each $t > 0$ exactly one of the players is the current leader and other N players are non-leaders. At time $t = 0$ all $N + 1$ bidders are non-leaders.

At each t, the non-leaders simultaneously choose whether to submit a bid or pass. Each submission of a bid costs C dollars and increases price by price increment ε. If $K > 0$ non-leaders submit a bid, each of them will be the leader in the next period with equal probability, $\frac{1}{K}$. So, if $K > 0$ bidders submit a bid at t, then $P_{t+1} = P_t + K\varepsilon$, and each of these K pays C dollars to the seller. The other non-leaders and the current leader will not pay anything at this round and will be non-leaders with certainty. The current leader cannot anything[15]. If all non-leaders pass at time t, the auction ends. If the auction ends[16] at $t = 0$, then the seller keeps the object and if it ends at $t > 0$, then the object is sold to the current leader at price P_t. Finally, if the game never ends, all bidders get payoffs $-\infty$ and the seller keeps the object. All the parameters of the game are commonly known and the players know the current leader and observe all the bids by all the players. We denote the actual revenue to the seller by R, expected revenue by $E(R)$, and expected revenue, conditional on object being

[14]We will assume that it includes also the delivery costs.

[15]This is a simplifying assumption. However, thinking about the practical auctions, it seems to be a plausible assumption to make. We will assume that the practical design of the auction is constructed so that whenever a current leader submitted a bid, the auctioneer or system assumes that it was just a mistake and ignores the bid.

[16]Note that the auction ends at $t > 0$ if and only if there has been positive number of bids, so $P_t > P_0$.

sold, by $E(R|\text{sale})$.

We will use the following normalizations. In case $\varepsilon > 0$, we normalize $v = \frac{V-P_0}{\varepsilon}$, $c = \frac{C}{\varepsilon}$, and $p_t = \frac{P_t-P_0}{\varepsilon}$, so that in particular $p_0 = 0$. In games where $\varepsilon = 0$ we use $v = V - P_0$, $c = C$, and $p_t = P_t - P_0$. Given the assumption and normalizations, a penny auction is fully characterized by (N, v, c, ε), where ε is only used to distinguish between infinite games discussed in Section 4 and finite games in Section 5.

Assumption 1. *We assume*[17] $v - c > \lfloor v - c \rfloor$ *and* $v > c + 1$.

The first assumption says that $v - c$ is not a natural number. It is just a technical assumption to avoid considering some tie-breaking cases, where the players are indifferent between submitting one last bid and not. The second assumption just ignores irrelevant cases, since $c + 1$ is the absolute minimum amount of money a player must spend to win the object. So, if the assumption does not hold, the game never starts.

As the solution concept we are considering Symmetric Stationary Subgame Perfect Nash Equilibrium (SSSPNE). We will discuss the formal details of this equilibrium in Appendix A and show that in the cases we consider SSSPNE are Subgame Perfect Nash Equilibria that satisfy two requirements. First property is Symmetry, which means that the players' identity does not play any role (so it could also be called Anonymity). The second property is Stationarity, which means that instead of conditioning their behavior on the whole histories of bids and identities of leaders, players only condition their behavior on the current price and number of active bidders.

In case $\varepsilon > 0$ this restriction means that we can use the current price p (independent on time or history how we arrived to it) as the current state variable and solve for a symmetric Nash equilibrium in this state, given the continuation values at states that follow each profile of actions. So the equilibrium is fully characterized by a $q : \{0, 1, \dots\} \to [0, 1]$, where $q(p)$ is the probability of submitting a bid that each non-leader independently uses at price p.

In case $\varepsilon = 0$ the equilibrium characterization is even simpler, since there are only two states. In the beginning of the game there is $N + 1$ non-leaders, and in any of the following histories the number of non-leaders is N. So, the equilibrium is characterized by $(\hat{q}_0, \hat{q})$ where $\hat{q}_0$ is the the probability that a player submits a bid at round 0 and $\hat{q}$ is the probability that a non-leader submits a bid at any of the following rounds. The SSSPNE can be found simply by solving for Nash equilibria at both states, taking into account the continuation values.

Lemmas 3, 4, and 5 in Appendix A show that any equilibria found in this way are SPNE satisfying Symmetry and Stationarity, and vice versa, any SSSPNE can be found using the described methods.

It must be noted that restricting the attention to this particular subset of Subgame Perfect Nash equilibrium, is restrictive and simplifies the analysis. As we will argue later, in general there are many other Subgame Perfect Nash equilibria in these auctions. The restrictions correspond to a situation where

[17] Where $\lfloor \cdot \rfloor$ is the floor function $\lfloor x \rfloor = \min\{k \in \mathbb{Z} : k \leq x\}$.

the players are only shown the current price. In practice players have more information, but in the case when they for one reason or another do not want to put in enough effort to keep track on all the bids (or believe that most of the opponents will not do it), the situation is similar. As an approximation this assumption should be quite plausible.

4 Auction with zero price increment

We will first look at a case where the price increment $\varepsilon = 0$. This is called "Free auction" (if $P_0 = 0$) or "Fixed-price auction" (if $P_0 > 0$) in Swoopo.com. One could also argue that this could be a reasonable approximation of a penny auction where ε is positive, but very small, so that the bidders perceive it as 0.

In this case the auction is an infinitely repeated game, since there is nothing that would bound the game at any round[18]. After each round of bids, bid costs are already sunk and the payoffs for winning are the same.

This is a well-defined game and we can look for SSSPNE in this game. Since the price does not change, the SSSPNE is fully characterized by a pair $(\hat{q}_0, \hat{q})$, where $\hat{q}_0$ is the probability that a non-leader will submit at round 0 and $\hat{q}$ the probability that a non-leader submits a bid at any round after 0. Let $\hat{v}^*, \hat{v}$ be the leader's and non-leaders' continuation values (after period 0).

Theorem 1. *In the case $\varepsilon = 0$, there exists a unique SSSPNE $(\hat{q}_0, \hat{q})$, such that*

1. *$\hat{q} \in (0, 1)$ is uniquely determined by equality $(1 - \hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$, where*[19]

$$\Psi_N(q) = \sum_{K=0}^{N-1} \binom{N-1}{K} q^K (1-q)^{N-(K+1)} \frac{1}{K+1}.$$

2. *If $N + 1 = 2$, then $\hat{q}_0 = 0$, otherwise $\hat{q}_0 \in (0, 1)$ is uniquely determined by $(1 - \hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v}$.*

Function $\Psi_N(q)$ is the player i's probability of becoming the new leader in after submitting a bid when $N - 1$ other non-leaders submit their bids independently, each with probability q. The $\frac{1}{K+1}$ part comes from the fact that if K other non-leaders submit bids, then i becomes the leader with this probability. Since each make their decision separately, K is Binomially distributed with parameters $(q, N - 1)$, which gives us the expression.

Proof. First notice that there is no pure strategy equilibria in this game, since if $\hat{q} = 1$, then the game never ends and all players get $-\infty$, which cannot be an equilibrium. Also, if $\hat{q} = 0$, then $\hat{v}^* = v$ and $\hat{v} = 0$. This cannot be an equilibrium, since a non-leader would want to deviate and submit a bid to get

[18] This is in contrast to $\varepsilon > 0$ case, where the game always ends in finite time. We will establish this in Lemma 1.

[19] $\binom{N}{K}$ is the binomial coefficient, $\binom{N}{K} = \frac{N!}{K!(N-K)!}, \forall 0 \leq K \leq N$.

$\hat{v}^* - c$, which is higher than $\hat{v}$, since $v > c + 1 > c$ by assumption. Therefore, in any equilibrium $\hat{q} \in (0, 1)$.

We will start with the case when $N + 1 = 2$. Since the equilibrium is in mixed strategies, non-leader's value must be equal when submitting a bid or not. If she submits a bid, she will be the next leader with certainty and the value of not submitting a bid is 0, since the game ends with certainty. Thus $\hat{v} = \hat{v}^* - c = 0$, and so $\hat{v}^* = c$. Being the leader, there is $(1 - \hat{q})$ probability of getting the object and $\hat{q}$ probability of getting $\hat{v} = 0$ in the next round, so $\hat{v}^* = (1 - \hat{q})v = c$ and therefore $\hat{q} = 1 - \frac{c}{v}$.

At $t = 0$, if $\hat{q}_0 > 0$ then expected value from bid is strictly negative[20], therefore the only possible equilibrium is such that $\hat{q}_0 = 0$, ie with no sale. This is indeed an equilibrium, since by submitting a bid alone gives $\hat{v}^* = c$ with certainty and costs c with certainty, so it is not profitable to deviate.

Note that when $N = 1$, then $\Psi_1(\hat{q}) = 1$, so $(1 - \hat{q})^N \Psi_1(\hat{q}) = 1 - \hat{q}$ and $(1 - \hat{q})v = \frac{c}{v} v = c = \hat{v}^*$, so the results are a special case of the claim from the theorem.

Suppose now that $N + 1 \geq 3$. Look at any round after 0. Again, this is a mixed strategy equilibrium, where $\hat{q} \in (0, 1)$, so non-leader's value is equal to the expected value from not submitting a bid. The other $N - 1$ non-leaders submit a bid each with probability $\hat{q}$, which means that the game ends with probability $(1 - \hat{q})^{N-1}$ and continues from the same point with probability $1 - (1 - \hat{q})^{N-1}$. Therefore

$$\hat{v} = [1 - (1 - \hat{q})^{N-1}]\hat{v} + (1 - \hat{q})^{N-1} 0 \iff \hat{v} = 0,$$

since $0 < \hat{q} < 1$. This gives the leader $(1 - \hat{q})^N$ chance to win the object and with the rest of the probability to become a non-leader who gets 0, so

$$\hat{v}^* = (1 - \hat{q})^N v + [1 - (1 - \hat{q})^N]\hat{v} = (1 - \hat{q})^N v.$$

The value of $\hat{q}$ is pinned down by the mixing condition of a non-leader

$$\hat{v} = 0 = \sum_{K=0}^{N-1} \binom{N-1}{K} \hat{q}^K (1 - \hat{q})^{N-1-K} \left[\frac{1}{K+1} \hat{v}^* + \frac{K}{K+1} \hat{v} \right] - c \iff$$

$$\frac{c}{v} = (1 - \hat{q})^N \sum_{K=0}^{N-1} \binom{N-1}{K} \frac{\hat{q}^K (1 - \hat{q})^{2N-(K+1)}}{K+1} = (1 - \hat{q})^N \Psi_N(\hat{q}).$$

By Lemma 6, $\Psi_N(\hat{q})$ is strictly decreasing continuous function with limits 1 and $\frac{1}{N}$ as $\hat{q} \to 0$ and $\hat{q} \to 1$ correspondingly. As $\hat{q}$ changes in $(0, 1)$, it takes all values in the interval $\left(\frac{1}{N}, 1\right)$, each value exactly once. Now, $(1 - \hat{q})^N$ is also strictly decreasing continuous function with limits 1 and 0, so the function $(1 - \hat{q})^N \Psi_N(q)$ is a strictly decreasing continuous function in $\hat{q}$ and takes all values in the interval $(0, 1)$. Since $0 < \frac{c}{v} < 1$ and there exists unique $\hat{q} \in (0, 1)$ that solves the equation $(1 - \hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$.

[20] The cost is certainly c, but expected benefit is weighted average c and 0 both with strictly positive probability.

Let us now consider period 0 to find the equilibrium strategy at $\hat{q}_0$. Denote the expected value that a player gets from playing the game by $\hat{v}_0$. We claim that $\hat{q}_0 \in (0,1)$. To see this, suppose first that $\hat{q}_0 = 0$, which means that the game ends instantly and all bidders get 0. By submitting a bid, a player could ensure becoming the leader with certainty in the next round and therefore getting value $\hat{v}^* - c = (1-\hat{q})^N v - c$. Equilibrium condition says that this must be less than equilibrium payoff 0, but then

$$(1-\hat{q})^N v - c \leq 0 \iff (1-\hat{q})^N \leq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}),$$

so $\Psi_N(\hat{q}) \geq 1$. This is contradiction, since $\Psi_N(\hat{q}) < 1$ for all $\hat{q} > 0$.

Suppose now that $\hat{q}_0 = 1$ is an equilibrium, so that each bidder must weakly prefer bidding to not bidding and getting continuation value of a non-leader, $\hat{v} = 0$. This gives equilibrium condition

$$\frac{1}{N+1}\hat{v}^* - c = \frac{1}{N+1}(1-\hat{q})^N v - c \geq 0 \iff \frac{(1-\hat{q})^N}{N+1} \geq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}),$$

so $\Psi_N(\hat{q}) \leq \frac{1}{N+1} < \frac{1}{N}$, which is a contradiction by Lemma 6.

Thus, in equilibrium $0 < \hat{q}_0 < 1$ is defined by

$$0 = \sum_{K=0}^{N} \binom{N}{K} \hat{q}_0^K (1-\hat{q}_0)^{N-K} \frac{1}{K+1}\hat{v}^* - c \quad \iff \quad (1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v}.$$

To show that this equation defines $\hat{q}_0$ uniquely (for a fixed $\hat{q} \in (0,1)$), we can rewrite it as follows.

$$(1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}) \quad \iff \quad \Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q}).$$

Now, $\Psi_N(\hat{q})$ is a fixed number. By Lemma 6 $\Psi_N(\hat{q}) \in \left(\frac{1}{N}, 1\right)$. As argued above (continuous, strictly decreasing) $\Psi_{N+1}(\hat{q}_0)$ takes values in the interval $\left(\frac{1}{N+1}, 1\right) \supset \left(\frac{1}{N}, 1\right)$, so the equation must have unique solution $\hat{q}_0$. $\square$

Corollary 1. *From Theorem 1 we get the following properties of the auctions with $\varepsilon = 0$:*

1. $\hat{q}_0 < \hat{q}$.
2. *If $N+1 > 2$, then the probability of selling the object is $1-(1-\hat{q}_0)^{N+1} > 0$. If $N+1 = 2$, the seller keeps the object.*
3. *Expected ex-ante value to the players is 0.*
4. *Expected revenue to the seller, conditional on sale, is v,*

Proof. We will prove each part and also give some intuition where applicable.

1. By Lemma 6, $\Psi_{N+1}(q) \geq \Psi_N(q)$. Since $\Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q})$ and $\Psi_K(q)$ is strictly decreasing function of q, we have $\hat{q}_0 < \hat{q}$.

   This is intuitive, since from the perspective of a non-leader, the two situations are identical in terms of continuation values, but at $t = 0$ there is one more opponent trying to become the leader.

2. This is just reading from the theorem. By the rules of the game, the seller sells the object whenever there was at least one bid, so the object is not sold only in the case when all bidders choose not to submit a bid at round 0. Therefore, the object is sold with probability $P(p > 0) = 1 - (1 - \hat{q}_0)^{N+1}$.

   If $N + 1 = 2$, then $\hat{q}_0 = 0$, so $P(p > 0) = 0$. If $N + 2 > 2$, then $\hat{q}_0 > 0$, so $P(p > 0) > 0$.

3. Let $\hat{v}_0$ be the expected ex ante value to the players. If $N + 1 = 2$, then $\hat{v}_0 = 0$, since players pass with certainty. If $N + 1 > 2$, then each bidder is at round 0 indifferent between bidding and not bidding, and not bidding gives 0 if none of the other players bid and $\hat{v} = 0$ of some bid. Therefore $\hat{v}_0 = 0$.

4. There is another way how the ex-ante value to the players, $\hat{v}_0$, can be computed. Let the actual number of bids the players submitted in a particular realization of uncertainty be B. Conditioning on sale means that $B > 0$.

   Since the value to the winner is v, and collectively all the players paid Bc in bid costs, the aggregate value to the players is $v - Bc$. By symmetry and risk-neutrality, ex-ante this value is divided equally among all players, so

$$0 = (N + 1)v(0) = \sum_{B=1}^{\infty} [v - Bc] P(B|B > 0) = v - cE(B|B > 0).$$

   Expected revenue to the seller, given that the object is sold, is Bc from all the bids. So

$$E(R|\text{sale}) = E(Bc|B > 0) = cE(B|B > 0) = v.$$

□

The following observations illustrate, that although in expected terms all the payoffs are precisely determined, in actual realizations almost anything can happen with positive probability.

Observation 1.

1. With probability $(N+1)(1-\hat{q}_0)^N \hat{q}_0 (1-\hat{q})^N > 0$ the seller sells the object after just one bid and gets $R = c$. The winner gets $v - c$ and the losers pay nothing.

2. When we fix arbitrarily high number $\bar{R}$, then there is positive probability that revenue $R > \bar{R}$. This is true since there is positive probability of sale and at each round there is positive probability that all non-leaders submit bids.

3. With positive probability we can even get a case where revenue is bigger than $\bar{R}$, but the winner paid just c.

Observation 2. None of the qualitative results in this case were dependent on the parameter values, so changes in parameters only affect the numerical outcomes.

1. In particular, given that the assumptions are satisfied, the expected revenue and the total payoff to the bidders does not depend on the parameter values other than the fact that $E(R|\text{sale}) = v$.

2. Equilibrium conditions were $(1-\hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$ and $\Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q})$ and functions $(1-\hat{q})^N \Psi_N(q)$, $\Psi_N(q)$, and $\Psi_{N+1}(q)$ are strictly decreasing. Therefore, as $\frac{c}{v}$ increases, both $\hat{q}$ and $\hat{q}_0$ will decrease.

   This means that for a fixed v, as c decreases, the probability of sale decreases. Note that in the limit as $c \to 0$, we get an auction that can be approximately interpreted as dynamic English auction. The puzzling fact is that in this auction the object is never sold.

3. As N increases, since $\Psi_N(q)$ is decreasing in N, both $\hat{q}$ and $\hat{q}_0$ decrease.

Remark 1. The discussion above was about SSSPNE. If we do not require stationarity and symmetry, then almost anything is possible in terms of equilibrium strategies, expected revenue to the seller, and the payoffs to the bidders. It is easy to see this from the following argument

1. Fix $i \in \{1, \ldots, N+1\}$. One possible SPNE is such that player i always bids and all the other players always pass. This is clearly an equilibrium since given i's strategy, any $j \neq i$ can never get the object and can never get more than 0 utility. Also, given that none of the opponents bid, i wants to bid, since $v - c > 0$. This equilibrium gives $v - c$ to i and 0 to all the other bidders.

2. Using this continuation strategy profile as a "punishment" we can construct other equilibria, including one where no-one bids (if i bids at the first round then some $j \neq i$ will punish him by always bidding in the next rounds that, so that the deviator i pays c and gets nothing, whereas punisher j will get $v - c > 0$).

3. Or we can construct an equilibrium where all the players bid $\lfloor v/c \rfloor$ times and then quit. If the bidding rule is constructed so that all bidders get non-negative expected value and are punished as described above, this is indeed a possible equilibrium. This will be the highest possible revenue from a pure strategy equilibrium with symmetry on the path of play.

4. With suitable randomizations it should be easy to construct equilibria that extract any revenue from c to v.

5 Auction with positive price increment

First we will introduce some additional notation. In a given equilibrium, let $q(p)$ be the non-leaders' probability to bid at price p. $v^*(p)$ denotes leader's value and $v(p)$ non-leaders' continuation values at price p. In a given equilibrium, $P(p)$ denotes the probability that price p will be the realized sales price. If $q(0) > 0$, $P(p|\text{sale}) = \frac{P(p)}{q(0)}$, is the distribution of the realization of sales price, conditional on object being sold.

$E(R|p > 0)$ denotes the expected revenue to the seller, conditional on the object being sold. R denotes the revenue under any realization of uncertainty.[21]

Finally, define $\tilde{p} = \lfloor v - c \rfloor$ and $\gamma = (v - c) - \lfloor v - c \rfloor \in [0, 1)$, so that $v = c + \tilde{p} + \gamma$. Note that by Assumption 1, $\gamma > 0$ and $\tilde{p} > 0$.

If price increment is positive and game goes on, the price rises. This means if the game does not end earlier, then sooner or later the price rises to a level where none of the bidders would want to bid. The following Lemma establishes this obvious fact formally and gives upper bound to the prices where bidders are still active.

Lemma 1. *Fix any equilibrium. None of the players will place bids at prices $p_t \geq \tilde{p}$. That is, $q(p) = 0$ for all $p \geq \tilde{p}$.*

Proof. First note that if $p > v$, then the upper bound of the winner's payoff in this game is $v - p < 0$ and therefore any continuation of this game is worse to all the players than end at this price. So, we know that the prices where $q(p) > 0$ are bounded by v.

Let $\hat{p}$ be the highest price where $q(\hat{p}) > 0$. Suppose by contradiction that $\hat{p} \geq \tilde{p} = \lfloor v - c \rfloor$. Since $q(\hat{p} + K) = 0$ for all $K \in \mathbb{N}$, the game ends instantly if arriving to these prices. Therefore $v(\hat{p} + K) = 0 < c$, and so

$$v^*(\hat{p} + K) = v - \hat{p} - K = (c + \tilde{p} + \gamma) - \hat{p} - K = \underbrace{\tilde{p} - \hat{p}}_{\leq 0} + \underbrace{\gamma - K}_{<0} + c < c,$$

So, if $K - 1 \in \{0, \ldots, N - 1\}$ opponents bid, by submitting a bid the agent gets strictly negative expected value. By not submitting a bid, any non-leader can ensure getting 0. Thus each non-leader has strictly dominating strategy not to bid at $\hat{p}$, which is a contradiction. Therefore $q(p) = 0$ for all $p \geq \tilde{p}$. □

Corollary 2. *With $\varepsilon > 0$, in any equilibrium:*

[21] As in the case of $\varepsilon = 0$, we have straightforward relation between these variables. If the actual sales price is p it means that p bids were made and so

$$R = (c + 1)p, \quad E(R|p > 0) = \sum_{p=1}^{\infty} (c + 1)pP(p|\text{sale}) = (c + 1)E(p|p > 0).$$

1. *Price level* $\max\{\tilde{p}-1+N, N+1\}$ *is an upper bound of the support of realized prices.*

   If $\tilde{p} > 1$*, then the last price where bidders could make bids with positive probability is* $\tilde{p}-1$ *and if all* N *non-leaders make bids, we will reach the price* $\tilde{p}+N-1$*. If* $\tilde{p} = 1$*, then the bidders only make bids at* 0 *and there are* $N+1$ *non-leaders at this stage, so the upper bound is* $N+1$*. Combination of these two cases gives us the upper bound.*

2. *The game is finite and there exists a a point of time* $\tau \leq \tilde{p}+N$*, where game has ended with certainty at any equilibrium. This is true since at each period when the game does not end, the price has to increase at least by* 1*.*

3. *All non-leaders have strictly dominating strategy not to bid at prices* $p_t \geq \tilde{p}$ *and at* $t+1$ *the game has ended with certainty. This means that we can use backwards induction to find any SPNE.*

5.1 Two-player case

The two-player case is very simple, since we have an alternating-move game, where at $t > 0$, one of the players is always leading and the other (non-leader) can choose whether to bid and become leader or pass and end the game. We can simply solve it by backwards induction. To see the intuition, let us start by solving a couple of backward induction steps before stating the result formally.

By Lemma 1, at prices $p \geq \tilde{p}$, the non-leader would never bid. Therefore, the continuation values values are $v^*(p) = v - p, v(p) = 0, \quad \forall p \geq \tilde{p}$, and in particular $v^*(\tilde{p}) = v - \tilde{p} = c + \gamma$.

At $p = \tilde{p}-1$, non-leader will make a bid since $v^*(p+1)-c = v^*(\tilde{p})-c = \gamma > 0$. Therefore $v^*(\tilde{p}-1) = v(\tilde{p}) = 0$, $v(\tilde{p}-1) = \gamma$.

At $p = \tilde{p}-2 > 0$, non-leader will not make a bid, since continuation value in the next round is 0 which does not cover the cost of bid. Thus $v^*(\tilde{p}-2) = v - (\tilde{p}-2) = c + \gamma + 2$, $v(\tilde{p}-2) = 0$.

We can continue this process for all $t > 0$ and then need to consider the simultaneous decision at stage 0. The following Proposition 1 characterizes the set of equilibria for two-player case.

Proposition 1. *Suppose* $\varepsilon > 0$ *and* $N+1 = 2$*. Then in any SSSPNE the strategies* q *are such that*

$$q(p) = \begin{cases} 0 & \forall p \geq \tilde{p} \text{ and } \forall p = \tilde{p} - 2i > 0, i \in \mathbb{N}, \\ 1 & \forall p = \tilde{p} - (2i+1) > 0, i \in \mathbb{N}, \end{cases}$$

and $q(0)$ *is determined for each* (v, c) *by one of the following cases.*

1. *If* $\tilde{p}$ *is an even integer, then* $q(0) = 0$*.*

2. *If* $\tilde{p}$ *is odd integer and* $v \geq 3(c+1)$*, then* $q(0) = 1$*.*

3. *If $\tilde{p}$ is odd integer and $v < 3(c+1)$, then $q(0) = 2\frac{v-(c+1)}{v+(c+1)} \in (0,1)$.*

Proof. As argued above, by Lemma 1, $q(p) = 0$ for all $p \geq \tilde{p}$. We want to show that

$$q(p) = \begin{cases} 0 & \forall p = \tilde{p} - 2i > 0, i \in \mathbb{N}, \\ 1 & \forall p = \tilde{p} - (2i+1) > 0, i \in \mathbb{N}. \end{cases}$$

We will show by mathematical induction that it holds for all $i \in \mathbb{N}$[22]. We already showed the induction basis for $i = 0$, since then $\tilde{p} - 2i = \tilde{p}$ and $\tilde{p} - (2i+1) = \tilde{p} - 1$.

Assuming that the claim is true for i, we want to show that it holds for $i+1$. Since $q(\tilde{p} - 2i) = 0$ the game ends and the leader wins instantly, so

$$v^*(\tilde{p} - 2i) = v - \tilde{p} + 2i = c + \gamma + 2i, \quad v(\tilde{p} - 2i) = 0.$$

Also, $q(\tilde{p} - (2i+1)) = 1$, that is, the price increases by 1 with certainty and the roles are reversed, so

$$v^*(\tilde{p} - (2i+1)) = v(\tilde{p} - 2i) = 0, \quad v(\tilde{p} - (2i+1)) = v - \tilde{p} + 2i - c = 2i + \gamma.$$

Let $p = \tilde{p} - 2(i+1)$. Then $p + 1 = \tilde{p} - (2i+1)$, so submitting a bid would give $v^*(\tilde{p} - (2i+1)) - c = -c$ to the non-leader, which is not profitable. Therefore $q(\tilde{p} - 2(i+1)) = 0$ and the leader gets

$$v^*(\tilde{p} - 2(i+1)) = v - \tilde{p} + 2(i+1) = c + \gamma + 2(i+1).$$

Let $p = \tilde{p} - (2(i+1)+1)$, so that $p + 1 = \tilde{p} - 2(i+1)$. Then making a bid would give $v^*(\tilde{p} - 2(i+1)) - c = \gamma + 2(i+1) > 0$ to the non-leader, which means that it is profitable to make a bid.

To complete the analysis, we have to consider $t = 0$, where $p = 0$ and both players are non-leaders simultaneously choosing to bid or not. In this stage, there three cases to consider.

First consider the case when $\tilde{p}$ is an even integer, ie $\tilde{p} = 2i+2$ for some $i \in \mathbb{N}$. Then $2 = \tilde{p} - 2i$ and $1 = \tilde{p} - (2i+1)$, so we get the strategic-form stage game in the Figure 5. In this game both players have strictly dominating strategy to

	B	P
B	$\frac{1}{2}(2i + \gamma - c), \frac{1}{2}(2i + \gamma - c)$	$-c, 2i + \gamma$
P	$2i + \gamma, -c$	$0, 0$

Figure 5: Period 0, case when $\tilde{p}$ is even

pass, ie $q(0) = 0$. That is, the unique SPNE in the case when $\tilde{p}$ is even, is the one where the seller keeps the object.

Suppose now that $\tilde{p}$ is odd number, ie $\tilde{p} = 2i+1$, so that $1 = \tilde{p} - 2i$ and $2 = \tilde{p} - (2i-1)$. Then we get strategic form in the Figure 6

[22] The claim specifies only $q(p)$ when $p > 0$, so actually we do not need to do it for all $i \in \mathbb{N}$, but it does not affect the proof.

	B	P
B	$\frac{\gamma}{2}+i-1-c, \frac{\gamma}{2}+i-1-c$	$2i+\gamma, 0$
P	$0, 2i+\gamma$	$0, 0$

Figure 6: Period 0, case when $\tilde{p}$ is odd

Note that $2i+\gamma = \tilde{p}-1+\gamma = v-c-1$, so $\frac{1}{2}(2i+\gamma-2)-c = \frac{1}{2}(v-3(c+1))$. The sign of this expression is not determined by assumptions, so we have to consider two cases.

If $v \geq 3(c+1)$, then bidding at round 0 is dominating strategy for both players, ie $q(0) = 1$. Both players will submit a bid at round 0, and the one who will be the non-leader will submit another bid after that. This means that in total players make 3 bids and the price ends up to be 3. This is where the condition $v \geq 3(c+1)$ comes from.

If $v < 3(c+1)$, then there is a symmetric MSNE[23], where both bidders bid with probability $q \in (0,1)$, where q is determined by

$$q\left(\frac{1}{2}(2i+\gamma-2)-c\right)+(1-q)(2i+\gamma) = 0 \iff$$

$$q(0) = \frac{2(2i+\gamma)}{2c+2+2i+\gamma} = 2\frac{v-(c+1)}{v+(c+1)} \in (0,1).$$

□

Observation 3. Some observations regarding the equilibria in the two-player case.

1. Equilibrium outcome are very sensitive to seemingly irrelevant detail — is $\tilde{p}$ even or odd.

2. Also, in a realistic case when $v \geq 3(c+1)$, the equilibrium collapses in a sense that $E(R|p>0) = 3(c+1)$, which can be much lower than v.

3. In a special case when $\tilde{p}$ is an odd integer and $v < 3(c+1)$, we get the results similar to $\varepsilon = 0$ case: $P(p>0) \in (0,1)$, $E(R|p>0) = v$, $v(0) = 0$.

   In this equilibrium the players submit bids with positive probabilities and hope that the other does not submit a bid. But if she does, players actually prefer to be non-leaders, since at price $p = 2$, non-leader submits one more bid and the game ends at $p = 3$. Therefore $P(0) > 0, P(1) > 0, P(2) = 0, P(3) > 0, P(p) = 0, \forall p \geq 4$.

[23]There are also two asymmetric pure-strategy NE in the subgame, (P,B) and (B,P), where one player makes exactly one bid, so the revenue is $c+1$ and the value for this bidder is $v-(c+1)$.

5.2 More than two players

In $N+1$-player case (for arbitrary $N \geq 2$) the discussion is similar to previous, but at each round we have 2 or more non-leaders choosing to bid or not simultaneously. To see how an equilibrium looks like, consider the Example 1.

Example 1. *Let $N+1=3, v=4.1, c=2$, and $\varepsilon>0$. The unique SSSPNE for this game is given in the Table 2. Since $q(0) \in (0,1)$, the expected utility for all*

p	$q(p)$	$v^*(p)$	$v(p)$	$P(p)$	$P(p\|p>0)$
0	0.2299		0	0.4567	
1	0.0645	2.7129	0	0.358	0.6588
2	0	2.1	0	0.1715	0.3157
3	0	1.1	0	0.0139	0.0255
4	0	0.1	0	0	0

Table 2: Example 1, solution

players is $v(0)=0$ and expected revenue for the seller $E(R|p>0)=v=4.1$.

Note that ex-ante expectation of the sales price is going to be non-trivial. In fact, with 2.5% probability we observe price 3, which implies revenue $3(2+1)=9$, which is significantly higher than 4.1. From this, $c+3=5>4.1=v$ is paid by the winner and both losers will pay 2.

By Lemma 1 in any game $q(p)=0, \forall p \geq \tilde{p}$. When we take $p=\tilde{p}+K$ for $K=0,1,\ldots$, then $q(\tilde{p}+K)=0$ and

$$v^*(\tilde{p}+K)=v-(\tilde{p}+K)=v-c-\tilde{p}+c-K=c+\gamma-K, \quad v(\tilde{p}+K)=0.$$

So, we can consider the rest of the game to be finite and solve it using backwards induction. Take $p \in \{0,\ldots,\tilde{p}-1\}$. If $p>0$, there are N non-leaders and if $p=0$, there are $N+1$. Denote the number of non-leaders by $\bar{N}$. Then one of the following three situations characterizes $q(p), v^*(p)$, and $v(p)$.

First, a stage-game equilibrium where all $\bar{N}$ non-leaders submit bids with certainty. In this case $q(p), v^*(p)$, and $v(p)$ are characterized by the three equalities in conditions (C1). This is an equilibrium if none of the non-leaders wants to pass and become non-leader at price $p+\bar{N}-1$ with certainty, which gives us the inequality condition in (C1).

Conditions 1 (C1)**.** *$q(p)=1$, $v^*(p)=v(p+\bar{N})$, and*

$$v(p)=\frac{1}{\bar{N}}v^*(p+\bar{N})+\frac{\bar{N}-1}{\bar{N}}v(p+N)-c \geq v(p+\bar{N}-1).$$

Secondly, there could be a stage-game equilibrium where all $\bar{N}$ non-leaders choose to pass. This is characterized by (C2).

Conditions 2 (C2)**.** *$q(p)=0$, $v^*(p)=v-p$, and $v(p)=0 \geq v^*(p+1)-c$.*

Finally, there could be a symmetric mixed-strategy stage-game equilibrium, where equilibrium, where all $\bar{N}$ non-leaders bid with probability $q \in (0,1)$. This gives us (C3).

Conditions 3 (C3)**.** *$0 < q(p) < 1$,*

$$v(p) = \sum_{K=0}^{\bar{N}-1} \binom{\bar{N}-1}{K} q^K (1-q)^{\bar{N}-1-K} \left[\frac{1}{K+1} v^*(p+K+1) + \frac{K}{K+1} v(p+K+1)\right] - c$$

$$= \sum_{K=1}^{\bar{N}-1} \binom{\bar{N}-1}{K} q^K (1-q)^{\bar{N}-1-K} v(p+K),$$

$$v^*(p) = (1-q)^{\bar{N}}(v-p) + \sum_{K=1}^{\bar{N}} \binom{\bar{N}}{K} q^K (1-q)^{\bar{N}-K} v(p+K).$$

Note that every equilibrium each $q(p)$ must satisfy either (C1), (C2), or (C3) and therefore an equilibrium is recursively characterized. However, nothing is saying that the equilibrium is unique. In Appendix C we have example, where at $p = 2$, each of the three sets of conditions gives different solutions and so there are three different equilibria. Moreover, in (C3) the equation characterizing q is $\bar{N} - 1$'th order polynomial, so it may have up to $\bar{N} - 1$ different solution which could lead to different equilibria.

Theorem 2. *In case $\varepsilon > 0$, there exists a SSSPNE $q : \mathbb{N} \to [0,1]$, such that q and the corresponding continuation value functions are recursively characterized (C1), (C2), or (C3) at each $p < \tilde{p}$ and $q(p) = 0$ for all $p \geq \tilde{p}$. The equilibrium is not in general unique.*

Proof. $N+1 = 2$ is already covered in Proposition 1 and is a very simple special case of the formulation above.

If $N + 1 > 2$, then the formulation above describes the method to find equilibrium q. The conditions (C1), (C2), and (C3) are written so that there are no profitable one-stage deviations. To prove the existence we only have to prove that there is at least one q that satisfies at least one of three sets of conditions.

At each stage, we have a finite symmetric strategic game. Nash (1951) Theorem 2 proves that it has at least one symmetric equilibrium[24]. Since there conditions are constructed so that any mixed or pure strategy stage-game Nash equilibrium would satisfy them, there exists at least one such q.

Finally, Appendix C gives a simple example where the equilibrium is not unique. □

[24]His concept of symmetry was more general — he showed that there is an equilibrium that is invariant under every automorphism (permutation of its pure strategies). Cheng, Reeves, Vorobeychik, and Wellman (2004) point out that in a finite symmetric game this is equivalent to saying that there is a mixed strategy equilibrium where all players play the same mixed strategy. They also offer a simpler proof for this special case as Theorem 4 in their paper.

Corollary 3. *With $\varepsilon > 0$, in any SSSPNE, we can say the following about $E(R|sale)$.*

1. *$E(R|sale) \leq v$,*
2. *if $q(p) < 1, \forall p$, then $E(R|sale) = v$,*
3. *In some games in some equilibria $E(R|sale) < v$.*

Proof.

1. Similarly to the proof of Corollary 1, the aggregate expected value to the players must be equal to v minus the aggregate payments, which is the sum of p and costs pc. The revenue to the seller is exactly the sum of all payments, so

$$(N+1)v(0) = v - E(p + pc|p > 0)v - E(R|\text{sale}).$$

Players' strategy space includes option of always passing, which gives 0 with certainty. Therefore in any SSSPNE, $v(0) \geq 0$, so $E(R|\text{sale}) \leq v$.

2. If $q(p) < 1$ for all p, then this mixed strategy puts strictly positive probability on the pure strategy where the player never bids. This pure strategy gives 0 with certainty and so $v(0) = 0$.

3. If $q(p) = 1$ for some p, then the previous argument does not work, since the player does not put positive probability on never-bidding pure strategy.

To prove the existence claim, it is sufficient to give an example. We already found in previous subsection that in $N + 1 = 2$ player case, if $\tilde{p}$ is odd and $v > 3(c + 1)$, then $q(0) = 1$ and $E(R|p > 0) = 3(c + 1) < v$. Example in Appendix C gives a more complex equilibrium (details are in the Table 3) where $q(0) \in (0, 1), q(1) = 0$, but $q(2) = 1$ and $E(R|p > 0) = 8.62 < 9.1 = v$.

□

The following lemma gives restriction how often the players can pass. It shows that there cannot be two adjacent price levels in $\{1, \dots, \tilde{p}\}$, where none of the bidders submits a bid. Lemma 1 showd that $\tilde{p}$ is the upper bound of the prices where bidders may submit bids. Lemma 2 says that at $\tilde{p} - 1$ players always bid with positive probability, so that it is the least upper bound.

Lemma 2. *With $\varepsilon > 0$, in any SSSPNE, $\nexists \hat{p} \in \{2, \dots, \tilde{p}\}$ st $q(\hat{p}-1) = q(\hat{p}) = 0$. In particular, $q(\tilde{p} - 1) > 0$.*

Proof. Suppose $\exists \hat{p} \in \{2, \dots, \tilde{p}\}$ such that $q(\hat{p} - 1) = q(\hat{p}) = 0$. Since $q(\hat{p}) = 0$, the game ends there with certainty and therefore $v^*(\hat{p}) = v - \hat{p}$.

$q(\hat{p} - 1) = 0$, so the game ends instantly and all non-leaders get 0. By submitting a bid at $\hat{p} - 1$ a non-leader would become leader at price $\hat{p}$ with certainty. So the equilibrium condition at $\hat{p} - 1$ is

$$0 \geq v^*(\hat{p}) - c = v - \hat{p} - c \iff \hat{p} \geq v - c = \tilde{p} + \gamma > \tilde{p}.$$

This is a contradiction with assumption that $\hat{p} \le \tilde{p}$. Since $q(\tilde{p}) = 0$ by Lemma 1, this also implies that $q(\tilde{p} - 1) > 0$. □

The following proposition says that, conditional on the object being sold, very high prices are reached with positive probability. In fact, with additional condition $\gamma < (N-1)c$, the upper bound of possible prices is reached with positive probability. This condition is easily satisfied in any real applications since $N \ge 2$ and in general the bid cost is higher than bid increment, so that $c > 1 > \gamma$.

Proposition 2. *Let $\varepsilon > 0$, fix any SSSPNE where the object is being sold with positive probability, and let p^* be the highest price reached with strictly probability. Then*

1. $\tilde{p} \le p^* \le \max\{\tilde{p} + N - 1, N + 1\}$,
2. *If $\gamma < (N-1)c$, then $p^* = \max\{\tilde{p} + N - 1, N + 1\}$.*

Proof.

1. By Corollary 2, $p^* \le \max\{\tilde{p} + N - 1, N + 1\}$.

   Since p^* is reached with positive probability and the higher prices are never reached, $q(p^*) = 0$. Equilibrium condition for this is $v^*(p^* + 1) - c \le 0$. When arriving to any $p > p^*$, the game ends with certainty, so in particular at $p^* + 1$ we have $v^*(p^* + 1) = v - p^* - 1$. This gives $p^* \ge v - c - 1 = \tilde{p} - (1 - \gamma) > \tilde{p} - 1$. Since p^* and $\tilde{p}$ are integers, this implies $p^* \ge \tilde{p}$.

2. First, when $\tilde{p} = 1$, then the last price where bidders submit bids is 0 and in this case they always do it with positive probability, so $p^* = N + 1 = \max\{\tilde{p} + N - 1, N + 1\}$.

   Suppose now that $\tilde{p} > 1$, so that $\max\{\tilde{p} + N - 1, N + 1\} = \tilde{p} + N - 1$. This can only be true if $P(p^* - N|\text{sale}) > 0$ and $q(p^* - N) > 0$.

   First, look at case $q(p^* - N) < 1$. This would mean that $P(\tilde{p} - 1|\text{sale}) > 0$ and by Lemma 2 $q(\tilde{p} - 1) > 0$, so $P(\tilde{p} - 1 + N|\text{sale}) > 0$, which is contradiction with $p^* < \tilde{p} - 1 + N$. Therefore $q(p^* - N) = 1$, so all non-leaders submit bids, knowing that all others do the same and the price rises to p^* with certainty. This can be an equilibrium action if

   $$\frac{1}{N} v^*(p^*) + \frac{N-1}{N} v(p^*) - c \ge v(p^* - 1).$$

   Since $p^* \ge \tilde{p}$, the game ends instantly at this price and therefore $v^*(p^*) = v - p^*$ and $v(p^*) = 0$. Finally, $v(p^* - 1) \ge 0$ (since player can always ensure at least 0 payoff by not bidding). This gives the condition

   $$v - Nc \ge p^* \ge \tilde{p} = v - c - \gamma \quad \Longleftrightarrow \quad \gamma \ge (N-1)c.$$

   This contradicts the assumption made in the proposition.

□

Corollary 4. *When the object is sold and condition $\gamma < (N-1)c$ is satisfied,*

1. *$R > v$ with positive probability,*
2. *$R < v$ with positive probability*

So, we have shown in previous Proposition that sometimes the object is sold at very high prices, and in this Corollary that sometimes the seller earns positive profits and sometimes incurs losses. This means that the auction has the stylized properties described in Section 2.

Proof.

1. By the previous proposition, there is positive probability that the object is sold at price $p^* \geq \tilde{p} + 1 = v - c + (1-\gamma)$. Therefore, when object is sold at price p^*, the revenue is

   $$R = (c+1)p^* \geq (c+1)[v - c + (1-\gamma)] > v \iff \frac{v}{c+1} > \frac{c-(1-\gamma)}{c},$$

   which holds as strict inequality, since $v > c+1$ and $\gamma < 1$.

2. Since $E(R|\text{sale}) \leq v$ and $R > v$ with strictly positive probability, it must be also $R < v$ with strictly positive probability.

□

With $\varepsilon > 0$ the equilibria are non-trivially related to parameter values. The number of equilibria may increase or decrease as parameter values changes, and the equilibrium outcomes may are generally affected non-monotonically. However, we can make some observations regarding the parameter values in the limits.

Observation 4.

1. If $c \to 0$, then in the limit we would get a version of Dynamic English auction. Perhaps contrary to the intuition this auction generally ends very soon. The general intuition of this observation is the following.

   Suppose $N+1 = 3$, $q(p+1) < 1$, and $q(p+2) < 1$; $v^*(p+1) > 0, v^*(p+2) > 0, v(p+1) = v(p+2) = 0$ and $c \to 0$. Then at price p there is certainly a stage-game equilibrium where $q = 1$ since $v^*(p+2) - c > v(p+1) = 0$. There are no equilibria $q < 1$, since player cannot be indifferent between positive expected value from bid and 0 from no bid. For this reason there will be relatively many prices where $q(p) = 1$. Now, if $q(p) = 1$ then being leader at p is in general worse than being non-leader, so at $p-1$ the players have lower incentives to bid. In many equilibria this leads to situation where $E(R|\text{sale}) \ll v$.

To put it in the other words, when cost of bid is small, then whenever there is positive expected value from bidding, players compete heavily, which drives down the value to the bidders and therefore there are low incentives to bid in earlier rounds.

2. If $c \to v - 1$, but still $c < v - 1$ (the upper bound for c), then the game gives positive utility to the bidders only if there is exactly one bid. $q = 0$ will not be an equilibrium, since lone bidder would get positive utility. Also, at $p > 0$ no-one bids. Therefore the unique equilibrium is such that $q(0)$ is a very small number and $q(p) = 0, \forall p > 0$. Then $E(R|\text{sale}) = v$, but probability of sale is very low.

   As mentioned above, if $c \geq v - 1$ or equivalently, $1 \geq v - c = \tilde{p} + \gamma$, then $\tilde{p} = 0$ and there can never be any bids. This is obvious, since to get positive payoff one needs to become a leader and minimal possible cost for this is $c + 1$.

3. Increase in v means that the game is getting longer and this means that there are more states with strategic decisions and generally more possible equilibria and non-trivial effect on strategies and revenue.

   Decrease in v has the opposite effect and as $v \to c + 1$ we get the case described above.

4. If N is very large, then $q(p) < 1$ for any p just because if $q(p) = 1$ this would mean that $p + N > v$ and so players cannot get positive value from bidding, whereas they have to incur cost and may ensure 0 by not bidding. Obviously, in $q(p)$ is not always 0, since it would still be good to be a lone bidder. So, in general we would expect to see many p's with low positive (and sometimes 0) values of $q(p)$. Since $q(p) < 1, \forall p$ we would have $E(R|\text{sale}) = v$.

6 Discussion

The model introduced in this paper has some interesting properties of penny auctions. In these auctions, the outcome to the individual bidders and to the seller is very unpredictable and varies in a large interval.

However, this kind of model is unable to replicate one property that the practical penny auctions seem to have. As shown in Figure 2, in real auctions the average profit margin seems to be significantly higher than zero. In penny auctions the objects sold have well-defined market value, handing over the object has alternative cost v to the seller. Since in the game above, the expected revenue is less than or equal to v, it means that the seller would always be better off by setting up a supermarket and selling the object at a posted price. This is not a property of the auction, but a general individual rationality argument — since individuals can always ensure at least 0 value by inactivity, it is impossible to extract on average more than the value they expect to get. To achieve an outcome where expected revenue is strictly higher than the value of the object, we would need to add something to the model.

A trivial way to overcome (or actually ignore) the problem is to say that the value to the seller is some $v_s < v = v_b$. It could be for example that the suggested retail value is by far higher than the cost to the seller, but around the value that the customers expect to get. This would obviously mean that there are expected profits, but it does not explain why the seller would not use alternative selling methods.

One explanation from practice seems to be that it is "Entertainment shopping". This could mean that the bidders get some positive utility from participating, some "Gambling value" v_g in addition to v if winning. Then again $v_b = v + v_g > v_s$. This could be true because winning an auction feels like an accomplishment. In this case this could be an increasing function of N (beating N opponents is great). There are other possible ways to model this entertainment value. (1) For example, modeling it as a lump-sum sum value just from participating or (2) as a positive income that is increasing in the number of bids. (3) Assuming that "Saving" money gives some additional happiness. Then instead of $v - p$ the player would have some increasing function $f(v - p)$. If it is linear, it is a simple transformation of previous.

Alternatively, individuals might not consider c to be at the same monetary scale as v and p, since it is partly sunk. In practice people buy "bid packs" with 50 or 100 bids at a time, so the story is complicated, but it is reasonable to think that with some probability an individual has marginal cost of next bid less than c. Suppose the bidders consider the cost of bid $c_b < c$. Then $E(R|\text{sale}) = (c+1)E(p|p > 0) > (c_b + 1)E(p|p > 0)$ and $0 \leq (N + 1)v(0) = v - (c_b + 1)E(p|p > 0)$, so it is possible to earn profit. There could be other ways to affect v, p, c via linear or lump-sum changes to tell other stories.

Another approach would be to consider some boundedly rational behavior or uncertainty in the model. A specific property of "penny auctions" seems to be that the price increase is marginal for a bidder. We could consider a case where individuals behave as (at least for a while) that the action is with $\varepsilon = 0$, but with value shrinking as the price increases. Generally this would not be an equilibrium in game-theoretic sense, but it might be realistic in practice and, as shown in this paper, is computationally easier, since there is always unique and explicitly characterized equilibrium.

Another question to consider is the reputation of players. Since in practical auctions the user name of a bidder is public, this could mean reputation effects between the auctions and during one auction. If a player has built a reputation of being "tough" bidder in previous auctions, since it is an all-pay auction, it obviously affects the other bidders. Then the first thing to notice is the fact that in this case the equilibrium is in general not symmetric. As we argued in some cases above, there could be (and in some cases are) equilibria, where one bidder always bids and other never bid. This means that there is reputation-type equilibrium even without any costs of reputation building, just some communication between bidders is enough. Of course, in the long run, it may be profitable to invest in building reputation and therefore there could be some types of behaviors to consider outside of our model.

Finally, in practice automated bids called Bid butlers are used. The system allows bidders to specify starting and ending prices and the number of bids the system should make on their behalf. Players can always cancel their Bid butler and the opponents see whether bid is made using Bid butler or manually. This may have interesting implications for the game. In most trivial way – just assuming the bidders can start and stop their Bid butlers at any moment of time, it would not affect the game at all, since everyone can replicate any strategy either with Bid butler or without. But when assuming there is some probability that the Bid butler is used while opponent is away from the game for some positive amount of time means that it may have reputation-type effect during the auction. By observing a bid by Bid butler, opponents update their belief about the next move slightly, and this may change their behavior radically.

As argued here, this is only the first attempt to characterize this type of auctions in a game-theoretic model. The next steps would involve adding some behavioral aspects that would probably benefit form a careful empirical analysis that would show which kind of behaviors or biases are behind of the outcomes that cannot be replicated by a straightforward model.

Appendices

A Symmetric Stationary Subgame Perfect Nash Equilibrium

We will now introduce formally the equilibrium concept used in this paper, Symmetric Stationary Subgame Perfect Nash Equilibrium (SSSPNE). Denote the vector of bids at round t by $b^t = (b_0^t, \ldots, b_N^t)$, where $b_i^t \in \{0, 1\}$ is 1 if player i submitted a bid at period t. Denote the leader after[25] round t by $l^t \in \{0, \ldots, N\}$. The information that each player has when making a choice at time t, or history at t, is $h^t = (b^0, l^0, b^1, l^1, \ldots, b^{t-1}, l^{t-1})$. The game sets some restrictions to the possible histories, in particular to become a leader, one must submit a bid, so $b_{l^t}^t = 1$, and the leader cannot submit a bid, $b_{l^{t-1}}^t = 0$, and h^t is defined only if none of the previous bid vectors b^τ is zero vector. Denote the set of all possible t-stage histories by $\mathcal{H}^t$, and the set of all possible histories, $\mathcal{H} = \bigcup_{t=0}^{\infty} \mathcal{H}^t$.

In this game, a pure strategy of player i is $b_i : \mathcal{H} \to \{0, 1\}$, where $b_i(h^t) = 1$ means that player submits a bid at h^t and 0 that the player passes. The game has perfect recall, so by Kuhn's theorem any mixed strategy profile can be replaced by an equivalent behavioral strategy profile $\sigma_i : \mathcal{H} \to [0, 1]$, such that $\sigma_i(h^t)$ is the probability that player i submits a bid at history h^t. Since it makes notation simpler, whenever we are talking about strategies in the text, we mean

[25] That is the non-leader that submitted a bid at t and became the leader by random draw

behavioral strategies. Note that by the rules of the game, at histories h^t where $l^t = i$, player i is the leader and can only pass.

Definition 1. A strategy profile σ is Symmetric if for all $t \in \{0, 1, \dots\}$, for all $i, \hat{i} \in \{0, \dots, N\}$, and for all $h^t = (b^\tau, l^\tau)_{\tau=0,\dots,t-1} \in \mathcal{H}^t$, if $\hat{h}^t = (\hat{b}^\tau, \hat{l}^\tau)_{\tau=0,\dots,t-1} \in \mathcal{H}^t$ satisfies

$$\hat{b}^\tau_j = \begin{cases} b^\tau_j & \forall j \notin \{i, \hat{i}\}, \\ b^\tau_i & j = \hat{i}, \\ b^\tau_{\hat{i}} & j = i, \end{cases} \qquad \hat{l}^\tau = \begin{cases} l^\tau & l^\tau \notin \{i, \hat{i}\}, \\ i & l^\tau = \hat{i}, \\ \hat{i} & l^\tau = i, \end{cases} \qquad \forall \tau = \{0, \dots, t-1\},$$

then $\sigma_i(\hat{h}^t) = \sigma_{\hat{i}}(h^t)$.

The Symmetry assumption simply states that when we replace the identities of two players, then nothing changes. This means that we could also call it Anonymity assumption. Intuitively, the assumption means that given that other N opponents make exactly the same choices and the uncertainty has realized the same way, different players would behave identically.

Let function L_i be the indicator function that tells whether player i is leader after history h^t or not,

$$L_i(h^t) = \mathbf{1}[i = l^t], \quad \forall i \in \{0, \dots, N\}, \forall h^t \in \mathcal{H}.$$

Let $\mathcal{S}$ be the set of states in the game and $S : \mathcal{H} \to \mathcal{S}$ the function mapping histories to states. In particular, we define these as

1. If $\varepsilon = 0$, then $\mathcal{S} = \{N + 1, N\}$, and

   $$S(h^t) = \begin{cases} N + 1 & h^t = \emptyset, \\ N & h^t \neq \emptyset. \end{cases}$$

   The reason: in infinite game the price does not increase, so the only thing players will condition their behavior is the number of active bidders, which is $N + 1$ in the beginning and N at any round after 0.

2. If $\varepsilon > 0$, then $\mathcal{S} = \{0, 1, \dots\}$, and

   $$S(h^t) = \sum_{\tau=0}^{t-1} \sum_{i=0}^{N} b^\tau_i.$$

   That is, the total number of bids made so far or equivalently, the normalized price p_t. Note that we do not have to explicitly consider two cases with two different numbers of players, since at $h^t = \emptyset$ we have $S(h^t) = 0$ and at any other history $S(h^t) > 0$.

Definition 2. A strategy profile σ is Stationary if for all $i \in \{0, \dots, N\}$, and for all pairs of histories $h^t = (b^\tau, l^\tau)_{\tau=0,\dots,t-1} \in \mathcal{H}$, $\hat{h}^{\hat{t}} = (\hat{b}^\tau, \hat{l}^\tau)_{\tau=0,\dots,\hat{t}-1} \in \mathcal{H}$ such that $L_i(h^t) = L_i(\hat{h}^{\hat{t}})$, and $S(h^t) = S(\hat{h}^{\hat{t}})$, we have $\sigma_i(h^t) = \sigma_i(\hat{h}^{\hat{t}})$.

Stationarity assumption means that the time and particular order of bids are irrelevant. The only two things that affect player's action are current state and the fact whether she is a leader or not.

Definition 3. SPNE strategy profile σ is Symmetric Stationary Subgame Perfect Nash Equilibrium SSSPNE if it is Symmetric and Stationary.

Lemma 3. *A strategy profile σ is Symmetric and Stationary if and only if it can be represented by $q : \mathcal{S} \to [0,1]$, where $q(s)$ is the probability bidder i bids at state $s \in \mathcal{S}$ for each non-leader $i \in \{0, \dots, N\}$.*

Proof. Since q is only defined on states S and equally for all non-leaders, it is obvious that it is a strategy profile that satisfies Symmetry and Stationarity, so sufficiency is trivially satisfied.

For necessity, take any strategy profile $\sigma = (\sigma_0, \dots, \sigma_N)$, where $\sigma_i : \mathcal{H} \to [0,1]$, that satisfies Symmetry and Stationarity. Construct functions $q_0, \dots, q_N$, where $q_i : S \to [0,1]$ by setting

$$q_i(S(h^t)) = \begin{cases} 0 & \forall h^t : L_i(h^t) = 1, \\ \sigma_i(h^t) & \forall h^t : L_i(h^t) = 0, \end{cases} \quad \forall h^t \in \mathcal{H}.$$

Our construction of S and Stationarity ensure that q_i is well-defined function.

We claim that adding Symmetry means that we get $q_i(s) = q(s)$ for all i and $s \in \mathcal{S}$. To see this, fix any i and h^t such that $s = S(h^t)$ and $L_i(h^t) = 0$. By construction, $q_i(s) = q_i(S(h^t)) = \sigma_i(h^t)$.

Now, fix any other non-leader, $\hat{i}$, so that $L_{\hat{i}}(h^t) = 0$. Construct another history $\hat{h}^t$ that is otherwise identical to h^t, but such that i and $\hat{i}$ are swapped. Then $S(\hat{h}^t) = s$ (obvious for both cases) and $L_{\hat{i}}(\hat{h}^t) = 0$. By Symmetry we have $\sigma_i(h^t) = \sigma_{\hat{i}}(\hat{h}^t)$. Therefore

$$q_{\hat{i}}(s) = q_{\hat{i}}(S(\hat{h}^t)) = \sigma_{\hat{i}}(\hat{h}^t) = \sigma_i(h^t) = q_i(s).$$

□

So, if strategy profile satisfies Stationarity and Symmetry, we can greatly simplify its representation. We can replace σ by q that is just defined for all $s \in \mathcal{S}$ instead of full set of histories $\mathcal{H}$. In the following two lemmas we show that at least in the cases considered in this paper the solution method is also simplified by these assumptions, since any SSSPNE can be found simply by solving for stage-game Nash equilibria for each state $s \in \mathcal{S}$ taking into account the solutions to other states and the implied continuation value functions.

Lemma 4. *With $\varepsilon > 0$, a strategy profile σ is SSSPNE if and only if it can be represented by $q : \mathcal{S} \to [0,1]$ where $q(s)$ is the Nash equilibrium in the stage-game at state s, taking into account the continuation values implied by transitions S.*

Proof. Necessity: If σ is SSSPNE, then by Lemma 3 it can be represented by q and since it is a SPNE, there cannot be profitable one-stage deviations.

Sufficiency: By Corollary 2 any auction with $\varepsilon > 0$ ends not later than $\tilde{p}+N$. So, although our game is (by the rules) infinite, it is equivalent in the sense of payoffs and equilibria with a game which is otherwise identical to our initial auction, but where after time $\tilde{p}+N$ the current leader gets the object at the current price. This is finite game and checking one-stage deviations is sufficient condition for SPNE. □

Lemma 5. *With $\varepsilon = 0$, a strategy profile σ is SSSPNE if and only if it can be represented by $q : \mathcal{S} \to [0, 1]$ where $q(s)$ is the Nash equilibrium in the stage-game at state s, taking into account the continuation values implied by transitions S.*

Proof. Necessity is identical to Lemma 4. Sufficiency:[26] Suppose q is Nash equilibrium in the stage-game equilibrium at each state s. To shorten the notation we will use the following notation: $\hat{q}_0 = q(N + 1)$, $\hat{q} = q(N)$, $\hat{v}_0$ is the continuation value of the game at state $N+1$, $\hat{v}$ is the continuation value of a non-leader and $\hat{v}^*$ is the continuation value of a leader at state N. By Theorem 1 we get $\hat{q} \in (0, 1)$, defined by $(1 - \hat{q})^N \Psi_N(\hat{q}) = \frac{c}{v}$, $\hat{q}_0 < 1$, $\hat{v} = 0$, and $\hat{v}^* = (1 - \hat{q})^N v$.

Take any history $h^t \neq \emptyset$ and individual i who is not the leader at h^t. Let σ_i be the strategy that ensure the highest expected value to player i at history h^t. Denote continuation value using σ_i at history h^τ by $V(h^\tau)$ for all h^τ following h^t. To shorten the notation, denote $\hat{V} = V(h^t)$. Suppose there exists profitable deviation at h^t. Then σ_i must also be profitable deviation and therefore $\hat{V} > \hat{v} = 0$.

Some of the histories h^{t+1} following h^t and i playing $\sigma_i(h^t)$ are such that i is a non-leader. In these situations all the other players use the same mixed strategy in all the continuation paths, so all payoff-relevant details are the same as at h^t. This means that at such histories h^{t+1}, it must be $V(h^{t+1}) = \hat{V}$. It cannot be higher, since $\hat{V}$ is maximum, and it can't be lower, since i could improve $V(h^t)$ by changing strategy starting from this h^{t+1}.

Other histories h^{t+1} following following $h^t, \sigma_i(h^t)$ are the ones where i is the leader. Being the leader at h^{t+1}, two things can happen to i's payoff. First, game may end at h^{t+1} and player i gets v. This happens with probability $(1 - \hat{q})^N$ as argued above. Secondly, i can become a non-leader at history h^{t+1} following h^{t+1}. For the same reason as above, $V(h^{t+2}) = \hat{V}$ for all such histories. Therefore in histories h^{t+1} where i is the leader,

$$V(h^{t+1}) = (1 - \hat{q})^N v + (1 - (1 - \hat{q})^N)\hat{V}.$$

The expected value at h^t is the expectation over all the continuation values $V(h^{t+1})$ following mixed action $\sigma_i(h^t)$ minus the expected bid cost. So, we can write

$$\hat{V} = V(h^t) = \sum_{h^{t+1}|h^t,\sigma_i(h^t)} P(h^{t+1}|h^t, \sigma_i(h^t))V(h^{t+1}) - c\sigma_i(h^t)$$

[26] Note that since the game does not satisfy continuity at infinity, checking one-stage deviations may not be sufficient for SPNE.

Using the values $V(h^{t+1})$ derived above and the fact that conditional on submitting a bid, the probability of becoming the leader at $t+1$ is $\Psi_N(\hat{q})$. So, the probability of become the leader is $\sigma_i(h^t)\Psi_N(\hat{q})$, which gives us

$$\begin{aligned}\hat{V} &= \sigma_i(h^t)\Psi_N(\hat{q})[(1-\hat{q})^N v + (1-(1-\hat{q})^N)\hat{V}] + [1-\sigma_i(h^t)\Psi_N(\hat{q})]\hat{V} - c\sigma_i(h^t) \\ &= c\sigma_i(h^t) + \sigma_i(h^t)\Psi_N(\hat{q})[1-(1-\hat{q})^N - 1]\hat{V} + \hat{V} - c\sigma_i(h^t) \iff \\ &\sigma_i(h^t)c\frac{\hat{V}}{v} = 0.\end{aligned}$$

By assumptions $c > 0$, $\hat{V} > 0$, and therefore $\sigma_i(h^t) = 0$. What we got is that by not bidding at h^t and at any following h^{t+1} and so on the player can ensure strictly positive expected payoff $\hat{V}$, which is impossible since the only way to get positive value is to be a leader and for this necessary condition is to bid. So there cannot be profitable deviations at any $h^t \neq \emptyset$.

We showed that at any history that follows h^0, always playing $\hat{q}$ ensures highest possible payoffs. Therefore at round 0 if there is profitable deviation, it must be one-stage deviation. But this is not possible, since we assumed that $\hat{q}_0$ is Nash equilibrium in the stage-game, taking into account the continuation values from $\hat{q}$ in the following periods. □

B Properties of $\Psi_N(q)$

The following Lemma helps us to characterize the set of equilibria and its properties in case when $\varepsilon = 0$. Let

$$\Psi_N(q) = \sum_{K=0}^{N-1} \binom{N-1}{K} q^K (1-q)^{N-1-K} \frac{1}{K+1}.$$

Lemma 6. *Let $N \geq 2$. Then*

1. *$\Psi_N(q)$ is strictly decreasing in $q \in (0,1)$.*
2. *$\lim_{q\to 0}\Psi_N(q) = 1$, $\lim_{q\to 1}\Psi_N(q) = \frac{1}{N}$.*
3. *$\Psi_N(q) > \Psi_{N+1}(q)$ for all $q \in (0,1)$.*

Proof.

1. $\Psi_N(q)$ is a differentiable function of q, so it is sufficient to show that $\frac{d\Psi_N(q)}{dq} < 0, \forall q \in (0,1)$. Differentiation and reordering of terms gives

$$\frac{d\Psi_N(q)}{dq} = \sum_{K=1}^{N-1} \frac{(N-1)!K}{(N-1-K)!(K+1)!} q^{K-1}(1-q)^{N-(K+1)}$$

$$-\sum_{K=0}^{N-2} \frac{(N-1)!(N-(K+1))}{(N-1-K)!(K+1)!} q^K (1-q)^{N-(K+1)-1}].$$

$$= \sum_{K=1}^{N-1} \frac{(N-1)!q^{K-1}(1-q)^{N-(K+1)}}{(N-1-K)!K!} \left[\frac{K}{K+1} - \frac{N-K}{N-K}\right].$$

$\frac{K}{K+1} < 1 = \frac{N-K}{N-K}$, so all terms in the sum are strictly negative for any $q \in (0,1)$.

2. Intuitively it is obvious: if other bidders almost certainly do not bid, probability of becoming the leader is close to 1, whereas when they almost certainly bid, each is becoming the leader with equal probability $\frac{1}{N}$.

   Formally, as $q \to 0$, all terms of $\Psi_N(q)$ where q is in positive power disappear, so only the one corresponding to $K = 0$ survives. Thus

$$\lim_{q\to 0} \Psi_N(q) = \lim_{q\to 0} \frac{(N-1)!}{(N-1)!0!} \frac{q^0(1-q)^{N-1}}{1} = 1.$$

   Similarly, as $q \to 1$, all terms where $(1-q)$ is in positive power disappear, so only the one where $K = N-1$ survives and we get

$$\lim_{q\to 1} \Psi_N(q) = \frac{(N-1)!}{0!(N-1)!} \frac{1}{N-1+1} = \frac{1}{N}.$$

3. Want to show that $\Delta_N(q) = \Psi_N(q) - \Psi_{N+1}(q) > 0, \forall q \in (0,1)$, where

$$\Delta_N(q) = \sum_{K=0}^{N-1} \frac{(N-1)!q^K(1-q)^{N-1-K}}{(N-K)!(K+1)!}(qN-K) - \frac{q^N}{N+1}.$$

   We prove it by first transforming the sum in a way that we get expectation of linear function over Binomial distribution with parameters $(q, N+1)$, since we know that expectation of the variable itself is $q(N+1)$ and expectation of constant is constant. The expression that remains after this manipulation depends only on q and N and is easy to analyze directly.

   First, change of variables in the sum, $L = K+1$

$$\Delta_N(q) = \sum_{L=1}^{N} \frac{(N-1)!q^{L-1}(1-q)^{N-L}}{(N+1-L)!L!}(qN+1-L) - \frac{q^N}{N+1}$$

$$= \frac{1}{N(N+1)q(1-q)} \sum_{L=1}^{N} \frac{(N+1)!q^L(1-q)^{N+1-L}}{(N+1-L)!L!}(qN+1-L) - \frac{q^N}{N+1}.$$

   Finally, we need to add and subtract terms with $L = 0$ and $L = N+1$

$$\Delta_N(q) = \frac{1}{N(N+1)q(1-q)} \sum_{L=0}^{N+1} \frac{(N+1)!q^L(1-q)^{N+1-L}}{(N+1-L)!L!}(qN+1-L)$$

$$-\frac{(1-q)^{N+1}(qN+1)}{N(N+1)q(1-q)} - \frac{-q^{N+1}(1-q)N}{N(N+1)q(1-q)} - \frac{q^N}{N+1}.$$

Using the properties of Binomial distribution and rewriting gives

$$\Delta_N(q) = \frac{qN + 1 - q(N+1)}{N(N+1)q(1-q)} - \frac{(1-q)^{N+1}(qN+1)}{N(N+1)q(1-q)}$$

$$= \frac{1 - (1-q)^N(qN+1)}{N(N+1)q}.$$

Therefore, to show that $\Delta_N(q) > 0$ for all $q \in (0,1)$, it is sufficient to show that $1 - (1-q)^N(qN+1) > 0$. Note that when $q = 0$ this expression is equal to 0, and it is strictly increasing in q

$$\frac{d}{dq}[1 - (1-q)^N(qN+1)] = qN(N+1)(1-q)^{N-1} > 0, \quad \forall q \in (0,1).$$

□

C A penny auction with multiple equilibria

Let $N + 1 = 3, v = 9.1, c = 2, \varepsilon > 0$. In this case, there are three SSSPNE, in Tables 3, 4, and 5 (which differ by actions at $p = 2$).

p	$q(p)$	$v^*(p)$	$v(p)$	$P(p)$	$P(p \mid p > 0)$
0	0.509		0	0.1183	
1	0	8.1	0	0.3681	0.4175
2	1	0	0	0	0
3	0.6996	0.5504	0	0.0119	0.0135
4	0	5.1	0	0.4371	0.4958
5	0.4287	1.3381	0	0.0211	0.0239
6	0.0645	2.7129	0	0.0277	0.0314
7	0	2.1	0	0.0157	0.0178
8	0	1.1	0	0.0001	0.0001
9	0	0.1	0	0	0

Table 3: Equilibrium with $q(2) = 1$

p	$q(p)$	$v^*(p)$	$v(p)$	$P(p)$	$P(p\|p>0)$
0	0.5266		0	0.1061	
1	0	8.1	0	0.354	0.3961
2	0.7249	0.5371	0	0.0298	0.0333
3	0.6996	0.5504	0	0.0273	0.0306
4	0	5.1	0	0.3344	0.3741
5	0.4287	1.3381	0	0.0484	0.0542
6	0.0645	2.7129	0	0.0636	0.0711
7	0	2.1	0	0.036	0.0403
8	0	1.1	0	0.0003	0.0003
9	0	0.1	0	0	0

Table 4: Equilibrium with $q(2) = 0.7249 \in (0,1)$

p	$q(p)$	$v^*(p)$	$v(p)$	$P(p)$	$P(p\|p>0)$
0	0		0	1	
1	0.7473	0.5174	0	0	
2	0	7.1	0	0	
3	0.6996	0.5504	0	0	
4	0	5.1	0	0	
5	0.4287	1.3381	0	0	
6	0.0645	2.7129	0	0	
7	0	2.1	0	0	
8	0	1.1	0	0	
9	0	0.1	0	0	

Table 5: Equilibrium with $q(2) = 0$

References

Baye, M. R., D. Kovenock, and C. G. de Vries (1996): "The all-pay auction with complete information," *Economic Theory*, 8(2), 291–305.

Cheng, S. F., D. M. Reeves, Y. Vorobeychik, and M. P. Wellman (2004): "Notes on equilibria in symmetric games," in *AAMAS-04 Workshop on Game-Theoretic and Decision-Theoretic Agents*, pp. 1–5.

Hendricks, K., A. Weiss, and C. Wilson (1988): "The War of Attrition in Continuous Time with Complete Information," *International Economic Review*, 29(4), 663–680.

Nash, J. (1951): "Non-Cooperative Games," *The Annals of Mathematics*, 54(2), 286–295.

Shubik, M. (1971): "The Dollar Auction game: a paradox in noncooperative behavior and escalation," *Journal of Conflict Resolution*, 15(1), 109–111.

Siegel, R. (2009): "All-Pay Contests," *Econometrica*, 77(1), 71–92.

Smith, J. M. (1974): "The theory of games and the evolution of animal conflicts.," *Journal of theoretical biology*, 47(1), 209–221.